

A WONDERFUL EMBEDDING OF THE LOOP GROUP

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ABSTRACT. I describe the wonderful compactification of loop groups. These compactifications are obtained by adding normal-crossing boundary divisors to the group LG of loops in a reductive group G (or more accurately, to the semi-direct product $\mathbb{C}^\times \ltimes LG$) in a manner equivariant for the left and right $\mathbb{C}^\times \ltimes LG$ -actions. The analogue for a torus group T is the theory of toric varieties; for an adjoint group G , this is the wonderful compactifications of De Concini and Procesi. The loop group analogue is suggested by work of Faltings in relation to the compactification of moduli of G -bundles over nodal curves. A thorough treatment of the construction of the ‘wonderful’ completion of the moduli stack of bundles will be carried out in a follow up paper.

1. INTRODUCTION

Let G be a simple simply connected algebraic group over \mathbb{C} . This paper studies smooth compactifications of G and limits of families of principal G bundles on nodal curves. The connection between the two can be seen as follows. If we fix a smooth curve C/\mathbb{C} then the stack $Bun_G(C)$ of all G -bundles on a curve satisfies the *valuative criterion for completeness*, which means given a commutative square as below we can always fill in the diagonal arrow:

$$\begin{array}{ccc} \mathrm{Spec} K & \longrightarrow & Bun_G(C) \\ \downarrow & \nearrow & \downarrow \\ \mathrm{Spec} R & \longrightarrow & \mathrm{Spec} \mathbb{C} \end{array}$$

where R is a complete DVR with fraction field K .

When C is a nodal curve then $Bun_G(C)$ no longer has this property. To see this let $E \rightarrow C$ be a principal G -bundle. We identify E with a bundle \tilde{E} on the normalization \tilde{X} together with an isomorphism $\phi: \tilde{E}_y \rightarrow \tilde{E}_z$ where y, z are the pre images of the node $x \in C$; we write $E = (\tilde{E}, \phi)$. $Bun_G(C)$ is not complete because in families the isomorphism ϕ can go to infinity along a 1PSG of G . Thus compactifications of G are relevant to any completion of $Bun_G(C)$.

In fact over a fixed nodal curve Bhosle in [2] has given a completion of $Bun_G(C)$ simply by supplying a compactification of G . However this does not address how to compactify bundles in families and does not give a modular interpretation of what the boundary of the completion means. Such a modular interpretation was given by Kausz for the case $G = GL(V)$. The key innovation we demonstrate here is that in order to provide a similar construction for G simple one must not just compactify G but in fact ‘compactify’ or complete the loop group LG .

Much work has been done in both the subject of compactifying reductive groups and the study of bundles on curves via loop groups. Let us give a brief account of some of the relevant results in these areas.

1.1. Compactifications of G . In 1983 De Concini and Procesi studied the symmetric space G/H where G is a Lie group and H is the fixed point set of an involution σ of G ; see [7]. They constructed, using representation theory, a “wonderful” compactification $\overline{G/H}$ of G/H . It is a smooth projective space and the boundary $\overline{G/H} - G/H$ consists of smooth normal crossing divisors. This construction was used to study problems in enumerative geometry. After De Concini and Procesi’s paper the properties of their compactification were axiomatized (with proper in place of projective) and such varieties were called *wonderful*. A particular case is $G = G \times G$ with $\sigma(g_1, g_2) = (g_2, g_1)$. We have $H = \Delta(G)$ so the

quotient is a copy of G . When G is of adjoint type this gives a wonderful compactification of G with the boundary being a union of $r = rk(G)$ smooth normal crossing divisors. A construction of the wonderful compactification which we will exploit uses representation theory:

$$\overline{G}_{wond} = \overline{G \times G[id]} \subset \mathbb{P}End(V(\lambda)) \quad (1)$$

where λ is a regular dominant weight.

In fact smooth compactifications \overline{G} for all reductive groups G exist [6, 6.2.4] but are not wonderful because they have too many boundary divisors; see definition of wonderful variety in section 2. Additionally there is a so called canonical embedding of a semi simple group but this compactification is generally not smooth unless we are in the extreme case $Z(G) = 1$ ($G = SL_2$ is an exception).

The issue with the canonical embedding can be understood as follows. The canonical embedding is always *toroidal* which means among other things that \overline{G} is smooth and complete if and only if \overline{T} is. The canonical embedding is in particular a spherical variety and, much like toric varieties, is classified by combinatorial data known as a colored fan. Colored fans are again just rational polyhedral fans with some decorations termed colors. An additional property of a toroidal embedding is that the underlying rational polyhedral fan for \overline{G} is equal to the fan of \overline{T} quotiented by the Weyl group of G . For the canonical embedding the fan is given by the Weyl chamber. Therefore the statement that \overline{G} is generally singular amounts to the statement that the roots of a reductive group G generally don't form a basis for the character lattice of a maximal torus $T \subset G$.

This problem can be circumvented by using stacks; specifically *toric stacks*; see [13]. It turns out that the generally singular fans that appear in these compactifications can be “enriched” to stacky fans which produce smooth DM-stacks. With toric stacks in hand one can construct a “spherical” G -stack \mathcal{X} that sits over the singular compactification \overline{G} . In [23], Martens and Thaddeus carry this out explicitly by constructing certain moduli problems about G -bundles on chains of \mathbb{P}^1 s that “represent” the compactification. We note that using a result from spherical varieties one can reproduce this result simply using the representation theory of G . Namely for a regular dominant weight λ there is a quasiprojective variety \overline{H}_c that sits inside endomorphism spaces associated to λ with an action of a torus T such that the global quotient $\mathcal{X} = [\overline{H}_c/T]$ contains G as a dense open subvariety. Additionally, \mathcal{X} contains a dense open substack \mathcal{X}_0 which is the closure of the open cell U^-TU of G and

Theorem 2.9.

- (a) \mathcal{X} is smooth and proper.
- (b) $\mathcal{X} - \mathcal{X}_0$ is of pure codimension 1 and we have an exact sequence

$$0 \rightarrow \mathbb{Z}^r \rightarrow Pic(\mathcal{X}) \rightarrow Z(G) \rightarrow 0$$

where the subgroup \mathbb{Z}^r is generated by the irreducible components of $\mathcal{X} - \mathcal{X}_0$.

- (c) The boundary $\mathcal{X} - G$ consists of r divisors D_1, \dots, D_r with simple normal crossings and the closure of the $G \times G$ -orbits are in bijective correspondence with subsets $I \subset [1, r]$ in such a way that to I we associate $\cap_{i \in I} D_i$.
- (d) Let u_1, \dots, u_r be generators of the rays of the Weyl chamber and M be the monoid they generate. Any G equivariant $\mathcal{X}' \rightarrow \mathcal{X}$ determines and is determined by a fan supported in the negative Weyl chamber whose lattice points lie in M .

Though this result is not new the fact that it can be proved using only representation theory will be important when we turn to the study of loop groups.

1.2. Loop groups. The definition of the loop group depends on what category one is working in. There is the smooth loop group: $L^{sm}K = C^\infty(S^1, K)$ where K is a compact form of G and its complexification $L^{sm}G = C^\infty(S^1, G)$; this is the version used in differential and complex geometry. In algebraic geometry one works with $LG = G((z)) = G(\text{Spec } \mathbb{C}((z)))$. In this paper we primarily work with the algebraic loop group but to a large extent results one proves in one setting tend to hold in the other. For example the geometry of flag varieties of LG and the representation theory are essentially the same for $L^{sm}G$. Here we state the main result in the algebraic case but we have an analogous result in the smooth setting (see

remark 12). In fact the group of interest is a semi-direct product $\mathbb{C}^\times \ltimes LG$; this is explained in the next subsection.

Theorem 3.7. *If G is a simple, connected, simply connected algebraic group over \mathbb{C} with center $Z(G)$ then $\mathbb{C}^\times \ltimes LG/Z(G)$ has a wonderful embedding $X^{aff} = \overline{\mathbb{C}^\times \ltimes LG/Z(G)}$. There is a dense open substack X_0^{aff} which is the closure of the open cell in $\mathbb{C}^\times \ltimes LG/Z(G)$ and*

- (a) X^{aff} is formally smooth.
- (b) $X^{aff} - X_0^{aff}$ is of pure codimension 1. It is a union of $r + 1$ divisors that generate the Picard group.
- (c) The boundary $X^{aff} - L^\times G/Z(G)$ consists of $r + 1$ normal crossing divisors D_1, \dots, D_r and the closure of the $G^{aff} \times G^{aff}$ -orbits are in bijective correspondence with subsets $I \subset [1, r + 1]$ in such a way that to I we associate $\cap_{i \in I} D_i$.
- (d) Any G equivariant $X' \rightarrow X^{aff}$ determine and is determined by a weyl equivariant morphism of toric varieties $\overline{T'} \rightarrow \overline{T_{ad,0}^\times}$.

We use the word embedding because $\overline{\mathbb{C}^\times \ltimes LG}$ does not quite satisfy the completeness criterion (see theorem 3.9 for a precise statement) but algebraically there is a ‘thin’ version of $\mathbb{C}^\times \ltimes LG$ which fibers over \mathbb{A}^1 and the boundary of the thin version is the fiber over 0 which itself is complete.

The strategy for proving the previous theorem is to use that $\mathbb{C}^\times \ltimes LG$ has a central extension G^{aff} which is a Kac-Moody group and thus has a well behaved representation theory. More precisely, we replace G with G^{aff} and $V(\lambda)$ with a highest weight representation $V(\tilde{\lambda})$ of G^{aff} . $V(\tilde{\lambda})$ is an infinite dimensional vector space which is a direct sum of weight spaces for a maximal torus $\mathbb{C}^\times \times T \times \mathbb{C}_c^\times$ of G^{aff} (here \mathbb{C}_c^\times is the central \mathbb{C}^\times).

Now we consider

$$X^{aff} = \overline{G^{aff} \times G^{aff} \cdot [id]} \subset \mathbb{P} \left[V(\tilde{\lambda}) \hat{\otimes} V(\tilde{\lambda})^* \right]$$

Where $V(\tilde{\lambda}) \hat{\otimes} V(\tilde{\lambda})^* := \prod_{\mu, \nu} \text{hom}(V_{\tilde{\mu}}, V_{\tilde{\nu}})$. We have $G^{aff}/Z(G^{aff}) = \mathbb{C}^\times \ltimes LG/Z(G)$ so X^{aff} indeed contains $\mathbb{C}^\times \ltimes LG/Z(G)$ as a dense open subscheme.

There is also a stacky extension analogous to theorem 2.9, see theorem 3.13.

1.3. LG and G -Bundles on Curves. The comparison between results related to LG and $L^{sm}G$ is not perfect. One notable example is that for any connected topological group G we have the stack $Bun_{G,hol}(C)$ of holomorphic principal G -bundles on C can be presented as

Theorem 1.1.

$$Bun_{G,hol}(C) = L_C^{sm}G \backslash L^{sm}G / L^{sm,+}G$$

Here L^+G consists of boundary values of holomorphic function from a small disc $D_p \cong \{z \in \mathbb{C} : |z| < 1\}$ and $L_C G$ consists of boundary values of holomorphic function $C - \overline{D_p} \rightarrow G$. The corresponding algebraic statement has $L^+G = G[[z]] = G(\text{Spec } \mathbb{C}[[z]])$ and $L_C G = G(C - p)$ and only holds for G semi simple.

The above theorem has a useful reformulation. Consider the two flag varieties $Y_+ = LG/L^+G$ and $Y_C = L_C G \backslash LG$ then

$$Bun_G(C) = \frac{Y_C \times Y_+}{LG}$$

The first presentation corresponds considering bundles on a small disc and on the complement of a small disc whereas the second corresponds removing a cylinder, or annulus, that connects the open curve to the small disc and gluing a bundle by a pair of transition functions at the two extreme ends of the cylinder.

We can use this second picture to understand degenerations. Holomorphically (or algebraically formally) any family of curves over a 1-dimensional base with smooth generic fiber and nodal special fiber has an étale neighborhood that looks like the genus 0 degeneration represented by the morphism

$$\text{Spec } \frac{\mathbb{C}[x, y, u]}{xy - u} \rightarrow \text{Spec } \mathbb{C}[u]$$

In the second picture, the space of bundles over the cylinder with trivializations at the ends is described using two loop groups $L_x G$ and $L_y G$ and the variables x, y are related by the relation $x = u/y$ for $u \neq 0$.

If we changed the trivialization at the x end by multiplication by $h(x) \in L_x G$ then this changes the trivialization at the y end by $h(u/y)$. So bundles on the cylinder can be identified with $L_x G \times L_y G / H$ where $H = \{(h(x), h(u/y))\}$. We can encode this with just one variable by looking at the semi direct product $\mathbb{C}^\times \ltimes LG$ where $u.\gamma(z).u^{-1} = \gamma(u.z)$. The subgroup H is isomorphic to $\Delta(LG)$ via $(h, h) \mapsto (h, uhu^{-1})$. Therefore away from $u = 0$, bundles on the cylinder are represented by

$$(\mathbb{C}^\times \ltimes LG) \times (\mathbb{C}^\times \ltimes LG) / \Delta(\mathbb{C}^\times \ltimes LG) \cong \mathbb{C}^\times \ltimes LG. \quad (2)$$

Much like when we considered limits in the group G for a fixed nodal curve; in families we now consider limits in the group $\mathbb{C}^\times \ltimes LG$. Using the embedding X^{aff} one can complete the moduli stack of bundles on nodal families as follows.

For simplicity assume C is a curve over a one dimensional base $C \rightarrow B$ with generic smooth fibers C_b and a single nodal fiber C_0 with a single node $x \in C_0$; the general case can be reduced to this case. $Bun_G(C)$ fibers over B with only one non complete fiber $Bun_G(C_0)$. To complete $Bun_G(C)$ it is enough to complete $Bun_G(C)$ in a neighborhood of C_0 and glue the result to $Bun_G(C - C_0)$.

In a neighborhood of C_0 we can use the ‘cylindrical’ slice Cyl mentioned above. That is $Cyl \subset C$ is an open subset and the fiber $Cyl_{b \neq 0}$ is a cylinder and Cyl_0 is two crossing discs that meet at $x \in C_0$. We can produce bundles on C by taking a bundle on $C - Cyl$ trivialized along its boundary and a bundle on Cyl trivialized along its boundary and gluing them with a ‘transition’ function in $LG \times LG$. Thus the non completeness of $Bun_G(C)$ has been confined to Cyl . Bundles on $Cyl - Cyl_0$ are represented by the homogenous space (2). Holomorphically we can express this as the homogeneous space

$$\frac{\mathbb{C}^\times \ltimes L^{sm}G \times \mathbb{C}^\times \ltimes L^{sm}G}{Hol(Cyl - Cyl_0, G)}$$

where $Hol(Cyl - Cyl_0, G)$ denotes holomorphic maps from $Cyl - Cyl_0$ to G . The embedding $\overline{\mathbb{C}^\times \ltimes LG}$ provides the necessary completion over this piece. The gluing mentioned in this construction posses no problems in the holomorphic setting but the appropriate algebraic analogue requires more care and will be addressed in a follow up paper.

1.4. Summary. Here we summarize briefly this paper. The aim of this paper is to construct an analogue of the wonderful compactification using positive energy representations. This is a generalization of the work in [29] to simple groups. In section 2, we recall the construction of the wonderful compactification of an adjoint group and use the theory of toric stacks to generalize to all reductive groups 2.9; this gives a representation theoretic construction to results in [23]. In section 3, we give basic definitions regarding loop groups and discuss positive energy representations. Additionally, we generalize the results in section 2 to the loop group setting. In section 4, we discuss in detail the positive energy representations of the loop groups LT for a torus T and describe the embeddings of LT in terms of combinatorics of positive definite forms. Additionally we describe highest weight representations of LG and give $\overline{L^\times G / Z(G)}$ the structure of an ind-scheme. In section 5, we discuss in more detail the connection with bundles on nodal curves and given an explicit example of an SO_5 bundle degenerating to a bundle with extra parabolic structure at the node. We also explain the connection between the embedding $\overline{L^\times G / Z(G)}$ and Faltings construction of a completion of $Bun_G(C)$.

2. DE CONCINI AND PROCECESI’S WONDERFUL COMPACTIFICATION

Here we give the construction of the wonderful compactification of a semisimple group of adjoint type, we recall the basic results regarding its structure and describe an extension to reductive groups. This section largely follows chapter 6 of [6].

Let G be a semisimple group. It has associated subgroups: a maximal torus T , opposite Borels B, B^- , their unipotent radicals U, U^- . The character lattice we denote as Λ_T , the co-character lattice we denote as V_T and if $\mu \in \Lambda_T, \eta \in V_T$ then the integer $\mu \circ \eta$ we denote as $\langle \mu, \eta \rangle$, $\langle \mu, \eta \rangle$, or $\mu(\eta)$. Let $r = rk(G)$ and let $\alpha_1, \dots, \alpha_r$ be the positive simple roots. Let $\omega_1, \dots, \omega_r$ be the fundamental weights; the monoid of dominant weights is denoted Λ_T^+ . In this section we focus on the adjoint group $G_{ad} := G/Z(G)$; here $Z(G)$ is the center of G .

For a reductive group H , define a normal H -variety Y to be *wonderful of rank r* if Y is smooth proper and has r normal crossing divisors D_1, \dots, D_r such that the H -orbit closures are given by intersections $\cap_{i \in I} D_i$ for any subset $I \subset \{1, \dots, r\}$. The group G_{ad} has a compactification \overline{G}_{ad} which is a $G_{ad} \times G_{ad}$ -wonderful variety of rank $r = rk(G_{ad})$.

One construction of \overline{G}_{ad} using representation theory is

$$X := \overline{G}_{ad} := \overline{G \times G \cdot [id]} \in \mathbb{P}End(V(\lambda)) = \mathbb{P}[V(\lambda) \otimes V(\lambda)^*] \quad (3)$$

where λ is a regular dominant weight, $V(\lambda)$ is the irreducible representation of highest weight λ , and $[id]$ is the class of the identity. The stabilizer in $G \times G$ of id is $\Delta(G)$ and the stabilizer of $[id]$ is $Z(G) \times Z(G) \cdot \Delta(G)$. So \overline{G}_{ad} indeed contains $G_{ad} = G/Z(G)$ as a dense open subset. We will see shortly that the construction is independent of the choice of λ .

Following [6] we set up the following notation. We can decompose

$$End(V(\lambda)) = \bigoplus_{\mu, \chi \in \Lambda_T} V_\mu \otimes V_\chi^*.$$

Set $\mathbb{P} = \mathbb{P}End(V(\lambda))$ and let $\mathbb{P}_0 \subset \mathbb{P}$ be the open subset consisting of points whose projection to $V_\lambda \otimes V_\lambda^*$ is not zero. Let \overline{T}_{ad} be the closure of T_{ad} in X . Set $X_0 = X \cap \mathbb{P}_0$ and $\overline{T}_0 = \overline{T} \cap \mathbb{P}_0$; we will see that X_0, \overline{T}_0 are independent of λ and this in turn implies the same for X .

Parts (a) - (d) of the following theorem are due to Deconcini and Procesi [7, Thm 3.1, Thm 7.6]. Statement (e) is proved in [6, 6.2.4].

Theorem 2.1. *Let $X = \overline{G}_{ad}$ be as in (3). Then*

- (a) *X is independent of λ .*
- (b) *X is smooth.*
- (c) *$X - X_0$ is of pure codimension 1; it consists of divisors that freely generate the Picard group.*
- (d) *The boundary $X - G_{ad}$ consists of r normal crossing divisors D_1, \dots, D_r and the closure of the $G \times G$ -orbits are in bijective correspondence with subsets $I \subset [1, r]$ in such a way that to I we associate $\cap_{i \in I} D_i$.*
- (e) *Any G equivariant $X' \rightarrow X$ determines and is determined by a Weyl equivariant morphism of toric varieties $T' \rightarrow \overline{T}$.*

One of the main results of this paper is an analogous theorem for loop groups. Certain results of semi simple group generalize immediately to loop groups and others are more subtle to prove. To illustrate this and to emphasize the main ideas involved we sketch a proof of the above theorem.

The proof will follow after a few lemmas which will also serve as our basic tools for studying stacky compactifications we construct for reductive groups and loop groups.

Let $t^{-\alpha}$ represent the regular function on T given by the character $-\alpha$.

Lemma 2.2. $\overline{T}_0 \cong \text{Spec } \mathbb{C}[t^{-\alpha_1}, \dots, t^{-\alpha_r}] \cong \mathbb{A}^r$

Proof. See the proof of 2.7. □

Lemma 2.3. *The action morphism $U^- \times U \times \overline{T} \rightarrow X$ sending $(u_1, u_2, t) \mapsto u_1 t u_2^{-1} \in X$ maps isomorphically onto X_0 .*

Proof. See 3.5. □

This lemma is a key tool in the proof of theorem 2.1; we shall refer to X_0 as the *open cell* of X .

Lemma 2.4. *Let G be a reductive group and let V a highest weight representation of G . Then the orbit of the highest weight is the only closed orbit in $\mathbb{P}V$*

Proof. This follows from the Borel fixed point theorem [3, Pg.272] for reductive groups. □

Remark 1. When we discuss loop groups we give a different proof (lemma 3.2) that circumvents the need for a Borel fixed point theorem in the Kac-Moody setting.

Let us now sketch a proof of theorem 2.1.

Proof. To prove (a), (b), we note that lemmas 2.2, 2.3 show \overline{T} and X_0 are independent of λ and smooth. Observe that

$$G \times G.X_0 = X \quad (4)$$

This follows from lemma 2.4: $G \times G.v_\lambda \otimes v_\lambda^*$ is the unique closed orbit in \mathbb{P} and it clearly intersects \mathbb{P}_0 . Now $\mathbb{P} - G \times G.\mathbb{P}_0$ is $G \times G$ -stable and closed therefore must be empty and hence we have (4). Consequently we have (b).

For (a), consider compactifications X_λ and X_μ associated to weights λ, μ . Let X_Δ be the closure of $\Delta(G)$ in $X_\lambda \times X_\mu$. The projection $p_\lambda: X_\Delta \rightarrow X_\lambda$ is equivariant and lemmas 2.2, 2.3 imply $X_{\Delta,0} := p_\lambda^{-1}(X_{\lambda,0}) \cong U^- \cdot \mathbb{A}^r \cdot U$; that is, the restriction of p_λ to $X_{\Delta,0}$ is an isomorphism and therefor induces an equivariant isomorphism on

$$\bigcup_{g \in G \times G} g.X_{\Delta,0} = X_\Delta \rightarrow X_\lambda = \bigcup_{g \in G \times G} g.X_0.$$

This proves (a).

Part (c) uses the result that if $U \subset Y$ is a dense open affine of any scheme Y then $Y - U$ is of pure codimension 1. This is proved in [25, 2.4]. In the case at hand, X_0 is affine and $\text{Pic}(X_0) = 0$; this immediately shows $\text{Pic}(X)$ is generated by the irreducible components of $X - X_0$. A relation among these generator is a principal divisor (f) which is invertible on X_0 ; such a function is a constant c and $f - c$ is zero on a dense open set hence there are no relations. Part (d) is proved by applying lemma 2.3 to reduce it to the case \overline{T}_0 for which it is obvious. Part (e) is addressed in the next section. \square

Remark 2. It is worth noting that there are alternative constuctions of the wonderful compactification. Starting with the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}$ consider the point $\Delta(\mathfrak{g}) \in Gr_{r,2r}$ and consider the closure of $G \times G.\Delta(\mathfrak{g})$ in $Gr_{r,2r}$; this gives another construction of the wonderful compactification of G ; see [7, pg.19].

Alternatively in [32], Vinberg constructs an affine scheme S with the action of $G \times T$ and identifies an open set $S^0 \subset S$ such that the GIT quotient $S^0//T$ is also isomorphic to the wonderful compactification. Thaddeus and Martens use this construction and generalize it to provide stacky compactifications of reductive groups; [23, § 5.6]. However in this paper we stick to methods of representation theory as they readily extend to loop groups.

2.1. Extension to Reductive Groups. Let G be a connected reductive group. In [6, 6.2], Brion and Kumar define a G -embedding Y to be a normal $G \times G$ variety containing $G = (G \times G)/\text{diag}(G)$ as an open orbit. They call X *toroidal* if the quotient map $G \rightarrow G/Z(G) = G_{ad}$ extends to a map $Y \rightarrow \overline{G_{ad}} =: X$ the wonderful compactification of G_{ad} . In fact, toroidal has a more general definition in the theory of spherical varieties but we will not need this level of generality.

They prove the following proposition; see [6, 6.2.4].

Proposition 2.5. *Any toroidal G embedding is determined by its associated toric variety.*

In particular, given reductive group G , a maximal torus T and a fan of the form $\Sigma = W \cdot \Sigma_0$ where W is the Weyl group and Σ_0 is a fan with support in the negative Weyl chamber, one can construct a G -embedding and an equivariant morphism to $\overline{G_{ad}}$. The construction in the proof of this result will be used in an essential way to generalize from smooth varieties to smooth stacks therefore we reproduce the proof and at the same time establish notation we will use in the sequel.

Proof. Let Y be a toroidal G -embedding. We have a map $\phi: Y \rightarrow \overline{G_{ad}}$ and the associated toric variety is $\phi^{-1}(\overline{T_{ad}})$. Equivariance implies that

$$\phi^{-1}(X_0) \cong U^- \times \phi^{-1}(\overline{T_{ad,0}}) \times U.$$

We now describe how to go from a Weyl equivariant toric variety to a G -embedding. The above observation will show these are inverse constructions.

Fix $\sigma \in \Sigma_0$ and let \overline{T}_σ be the corresponding toric variety. The coordinate ring is $k[\overline{T}_\sigma] = k[\sigma^\vee \cap \Lambda_T]$ and is generated by a finite set F_σ of characters. Now we choose a regular dominant weight λ such that $\lambda + F_\sigma$ are all regular dominant weights. For any set of regular dominant weights ν_1, \dots, ν_i , let

$\mathbb{P}[\nu_1, \dots, \nu_i]$ denote $\mathbb{P}(\bigoplus_i \text{End}(V(\nu_i)))$. As in the construction of $\overline{G_{ad}}$ we consider the orbit of the identity in $\mathbb{P}[\lambda, \lambda + F_\sigma] = \mathbb{P}[\lambda, \lambda + f]_{f \in F_\sigma}$. Consider the open set

$$W(\lambda, \sigma) = \mathbb{P}[\lambda, \lambda + F_\sigma] - \mathbb{P}[\lambda + F_\sigma]$$

where $\mathbb{P}[\lambda + F_\sigma]$ is embedded in the larger projective space by setting all coordinates in $\mathbb{P}[\lambda]$ to zero. Let $p_\sigma: W(\lambda, \sigma) \rightarrow \mathbb{P}[\lambda]$ be the projection. Set $\overline{G}_\sigma = p_\sigma^{-1}(\overline{G_{ad}})$. Note that p_σ is G -equivariant hence \overline{G}_σ is $G \times G$ stable and in particular is a G -embedding. In summary, we have a commutative diagram

$$\begin{array}{ccc} \overline{G}_\sigma & \hookrightarrow & W(\lambda, \sigma) \\ \downarrow p_\sigma & & \downarrow \\ \overline{G_{ad}} & \hookrightarrow & \mathbb{P}[\lambda] \end{array} \quad (5)$$

\overline{G}_σ is a quasiprojective G -embedding. Note that arguing as in the case of the wonderful compactification using lemma 2.3 and lemma 2.7 below we conclude that the partial compactifications \overline{G}_σ are independent of the choice of λ and on the choice of generators F_σ . Now it remains to check that the various \overline{G}_σ as σ ranges in Σ_0 glue together. This is the content of 2.6. \square

As in the case of $\overline{G_{ad}}$ we refer to $p_\sigma^{-1}(X_0)$ as the open cell of \overline{G}_σ . Using lemma 2.7 we reduce the gluing construction to the torus: for any cone $\sigma \in \Sigma_0$ let F_σ denote a choice for a set of generators for σ^\vee . If $\tau \subset \sigma$ then we can choose $F_\sigma \subset F_\tau$. Let λ_τ be such that $\lambda_\tau + F_\tau$ are all regular dominant weights. There is a projection $W(\lambda_\tau, \tau) \rightarrow W(\lambda_\tau, \sigma)$ which maps $\overline{G}_\tau \rightarrow \overline{G}_\sigma$ and using lemma 2.7, equation (4) and the fact that $\overline{T}_\tau \rightarrow \overline{T}_\sigma$ is an open immersion we conclude the same for $\overline{G}_\tau \rightarrow \overline{G}_\sigma$. If we have $\sigma_1 \supset \tau \subset \sigma_2$ then we get open immersions

$$\overline{G}_{\sigma_1} \xleftarrow{f_1} \overline{G}_\tau \xrightarrow{f_2} \overline{G}_{\sigma_2}$$

which we glue along. Now let us prove

Lemma 2.6. *The maps $\phi_f: \text{im } f_1 \rightarrow \text{im } f_2$ given by $\phi_f(f_1(p)) = f_2(p)$ satisfy the co-cycle condition*

Proof. Let $\sigma_1, \sigma_2, \sigma_3$ be cones in Σ_0 and set $\tau_1 = \sigma_1 \cap \sigma_2$, $\tau_2 = \sigma_2 \cap \sigma_3$, $\tau_3 = \sigma_3 \cap \sigma_1$ and $\rho = \sigma_1 \cap \sigma_2 \cap \sigma_3$. We can form the following pull back diagram

$$\begin{array}{ccc} \overline{G}_{\tau_3} \times_{\overline{G}_{\sigma_1}} \overline{G}_{\tau_1} & \longrightarrow & \overline{G}_{\tau_1} \\ \downarrow & & \downarrow \\ \overline{G}_{\tau_3} & \longrightarrow & \overline{G}_{\sigma_1} \end{array}$$

Roughly, $\overline{G}_{1,23} := \overline{G}_{\tau_3} \times_{\overline{G}_{\sigma_1}} \overline{G}_{\tau_1}$ should be thought of as the image of “ $\overline{G}_{\sigma_1} \cap \overline{G}_{\sigma_2} \cap \overline{G}_{\sigma_3}'' \rightarrow \overline{G}_{\sigma_1}$. Similarly we have spaces $\overline{G}_{2,13} := \overline{G}_{\tau_1} \times_{\overline{G}_{\sigma_2}} \overline{G}_{\tau_2}$ and $\overline{G}_{3,12} := \overline{G}_{\tau_2} \times_{\overline{G}_{\sigma_3}} \overline{G}_{\tau_3}$. It is routine to see that the co-cycle condition amounts to isomorphisms

$$\overline{G}_\rho \cong \overline{G}_{i,jk}$$

and the following diagram commuting.

$$\begin{array}{ccccc} & & \overline{G}_\rho & & \\ & \swarrow & \downarrow & \searrow & \\ \overline{G}_{\tau_1} & & \overline{G}_{\tau_2} & & \overline{G}_{\tau_3} \\ \swarrow f_1 \quad \searrow f_2 & & \swarrow g_1 \quad \searrow g_2 & & \swarrow h_1 \quad \searrow h_2 \\ \overline{G}_{\sigma_1} & & \overline{G}_{\sigma_2} & & \overline{G}_{\sigma_3} & & \overline{G}_{\sigma_1} \end{array} \quad (6)$$

Because we are dealing with toroidal embeddings both of these claims reduce to the torus for which it is obvious. To illustrate, if $p \in \overline{G}_{1,23}$ then we can write $p = (p_1, p_2)$ where both points map to the same

point $q \in \overline{G}_{\sigma_1}$. After translating by an element of $G \times G$ we can assume $q \in \overline{T}_{\sigma_1}$ and similarly for the p_i thus p represents a point in $\overline{T}_{\tau_3} \times_{\overline{T}_{\sigma_1}} \overline{T}_{\tau_1} \cong \overline{T}_{\rho}$. \square

Lemma 2.7. *In the notation of (5), we have $p_{\sigma}^{-1}(X_0) \cong U^{-} \times \overline{T}_{\sigma} \times U$ and the closure of the torus in $T \subset \overline{G}_{\sigma}$ is given by the fan whose cones are $w\sigma$ for $w \in W$ and their faces.*

Proof. The restriction of $p_{\sigma}: \overline{G}_{\sigma} \rightarrow X = \overline{G}_{ad}$ to G is just the projection $G \rightarrow G_{ad}$. As U^{\pm} do not intersect the center of G we have $p_{\sigma}(U^{\pm}) = U^{\pm}$. From the equivariance of p_{σ} and from $X_0 \cong U^{-} \times \overline{T}_0 \times U$ we have

$$p_{\sigma}^{-1}(X_0) = U^{-} \times p_{\sigma}^{-1}(\overline{T}_0) \times U$$

So we are reduced to showing $p_{\sigma}^{-1}(\overline{T}_0) = \overline{T}_{\sigma}$. Let μ_1, \dots, μ_m be generators for σ^{\vee} . The toric variety in question is the closure of the image of T in $\mathbb{P}[\lambda, \lambda + \mu_i]$; we can depict the image as:

$$\left[\begin{array}{cccc} \lambda(t) & & & \\ & \frac{\lambda(t)}{\alpha_1(t)} & & \\ & & \frac{\lambda(t)}{\alpha_2(t)} & \\ & & & \ddots \end{array} \right] \oplus \left[\begin{array}{cccc} \lambda(t)\mu_1(t) & & & \\ & \frac{\lambda(t)\mu_1(t)}{\alpha_1(t)} & & \\ & & \frac{\lambda(t)\mu_1(t)}{\alpha_2(t)} & \\ & & & \ddots \end{array} \right] \oplus \dots \oplus \left[\begin{array}{cccc} \lambda(t)\mu_m(t) & & & \\ & \frac{\lambda(t)\mu_m(t)}{\alpha_1(t)} & & \\ & & \frac{\lambda(t)\mu_m(t)}{\alpha_2(t)} & \\ & & & \ddots \end{array} \right]$$

The primage of \overline{T}_0 consists of those points where $\lambda(t) \neq 0$. This is:

$$\left[\begin{array}{cccc} 1 & & & \\ & \alpha_1^{-1}(t) & & \\ & & \alpha_2^{-1}(t) & \\ & & & \ddots \end{array} \right] \oplus \left[\begin{array}{cccc} \mu_1(t) & & & \\ & \frac{\mu_1(t)}{\alpha_1(t)} & & \\ & & \frac{\mu_1(t)}{\alpha_2(t)} & \\ & & & \ddots \end{array} \right] \oplus \dots \oplus \left[\begin{array}{cccc} \mu_m(t) & & & \\ & \frac{\mu_m(t)}{\alpha_1(t)} & & \\ & & \frac{\mu_m(t)}{\alpha_2(t)} & \\ & & & \ddots \end{array} \right]$$

Let σ_W denote the negative Weyl chamber in V_T . Recall we require $\sigma \subset \sigma_W$ so all $-\alpha_i \in \sigma_W^{\vee} \subset \sigma^{\vee}$. It follows that all the diagonal entries are polynomials in μ_1, \dots, μ_m so the projection to $\text{Spec } k[\mu_1(t), \dots, \mu_m(t)] \cong \overline{T}_{\sigma}$ is an isomorphism. The last statement of the proposition follows because $\overline{T}_{\sigma} = p_{\sigma}^{-1}(\overline{T}_{ad})$ and

$$\overline{T}_{ad} = \bigcup_{w \in W} w \overline{T}_{ad,0} w^{-1}.$$

This in turn follows from the $G \times G$ equivariance of the compactification \overline{G}_{ad} which gives $W \times W$ equivariance of \overline{T}_{ad} . \square

2.2. The Construction. In general the fans Σ that arise for reductive groups produce singular toric varieties and hence singular G -embeddings. However these toric varieties are always smooth as stacks. Here we describe a modification of the above construction that produces a smooth stack. We briefly recall the basic theory of toric stacks and then incorporate them into the above construction.

2.3. Preliminaries on Toric Stacks. Following [13], we define *toric stacks* as $[Y(\Sigma)/Z]$ where $Y(\Sigma)$ is a normal toric variety with associated fan Σ and Z is a subgroup of the torus $T_{\Sigma} \subset Y(\Sigma)$. The stack $[Y(\Sigma)/Z]$ contains the torus $T = T_{\Sigma}/Z$ as a dense open subscheme. Just as in the theory of toric varieties, toric stacks are encoded by combinatorial data called stacky fans. A *stacky fan* is a pair $(\Sigma, \beta: L \rightarrow N)$ where L, N are lattices, Σ is a fan in $L \otimes \mathbb{R}$, and β is a homomorphism of finite index.

The equivalence between toric stacks and stacky fans is given as follows. Given $[Y(\Sigma)/Z]$ we get a surjection $T_{\Sigma} \rightarrow T$ which induces a map $\beta: V_{T_{\Sigma}} \rightarrow V_T$. Thus we get the stacky fan

$$(\Sigma, \beta: V_{T_{\Sigma}} \rightarrow V_T).$$

Starting from $(\Sigma, \beta: N \rightarrow L)$ we note that the hypothesis on β implies that the dual morphism

$$\beta^*: \text{hom}(L, \mathbb{Z}) = L^* \rightarrow N^* = \text{hom}(N, \mathbb{Z})$$

is injective. Consider the tori $T_{\Sigma} := \text{hom}(N^*, \mathbb{C}^{\times})$ and $T := \text{hom}(L^*, \mathbb{C}^{\times})$. Dualizing β^* we get a surjection

$$0 \rightarrow Z(\beta) \rightarrow T_{\Sigma} \xrightarrow{\text{hom}(\beta^*, \mathbb{C}^{\times})} T \rightarrow 0$$

and thus we get the toric stack $[Y(\Sigma)/Z(\beta)]$.

Here is how toric stacks appear in our situation. Recall the notation $G, T, \alpha_1, \dots, \alpha_r$ from section 2. The Weyl chamber:

$$C = \{v \in V_T \otimes \mathbb{R} \mid \alpha_i(v) \geq 0\}$$

is a fundamental domain for the action of the Weyl group on $V_T \otimes \mathbb{R}$ and defines a rationally smooth fan. Let u_1, \dots, u_r be generators of the rays of the fan C . We get a homomorphism of lattices

$$\beta: \mathbb{Z}^r \xrightarrow{e_i \mapsto u_i} V_T$$

and let c be the standard cone generated by the coordinate rays in $\mathbb{Z}^r \otimes \mathbb{R}$ giving rise to the toric variety \mathbb{A}^r . We have that $\beta(c)$ generates the fan associated to the Weyl chamber. Consequently we get a smooth toric stack $[\mathbb{A}^r/Z(\beta)]$ associated to the stacky fan (c, β) whose coarse moduli space is the toric variety associated to the Weyl chamber in $V_T \otimes \mathbb{R}$. For general toric stacks the subgroup $Z(\beta)$ is arbitrary but in the present situation $Z(\beta)$ is always finite. In what follows we use the isomorphism $[\mathbb{A}^r/Z(\beta)] \cong [(\mathbb{A}^r \times T)/(\mathbb{C}^\times)^r]$ where $(\mathbb{C}^\times)^r$ acts diagonally on the product.

2.4. Stacky toroidal G -embeddings. Consider the reductive group $H = G \times (\mathbb{C}^\times)^r$ where G is semi simple. The Weyl chamber of H is $C \oplus \mathbb{Z}^r$ where C is the Weyl chamber of G . Consider the cone $c = \mathbb{N}^r = (\vec{0}, \mathbb{N}^r) \subset -C \oplus \mathbb{Z}^r$; it is a cone with support in the negative Weyl chamber. The dual cone c^\vee is generated by the fundamental weights $\pm\omega_i$ and e_i where $e_i: (\mathbb{C}^\times)^r \rightarrow \mathbb{C}^\times$ is the i th projection. By 2.5 we get an H -embedding

$$\overline{H}_c \subset \mathbb{P}[\lambda, \lambda \pm \omega_i, \lambda + e_i]$$

From the stacky fan (c, β) of the previous section we get a surjection

$$0 \rightarrow Z(\beta) \rightarrow (\mathbb{C}^\times)^r \xrightarrow{\pi} T \rightarrow 0,$$

thus we have a homomorphism

$$T_\beta := (\mathbb{C}^\times)^r \xrightarrow{\pi, id} T \times (\mathbb{C}^\times)^r \subset H.$$

The compactification \overline{H}_c carries an action of $H \times H$ so we get an action of $T_\beta \times T_\beta$. Identify T_β with $T_\beta \times T_\beta / \Delta(T_\beta)$.

Definition 2.8. The stacky wonderful compactification of G is

$$\mathcal{X} = \overline{G} := [\overline{H}_c / T_\beta]$$

Over the open cell in \overline{H}_c this stack quotient is

$$\begin{aligned} [(U^- \times T \times \mathbb{A}^r \times U) / T_\beta] &\cong U^- \times [(T \times \mathbb{A}^r) / T_\beta] \times U \\ &\cong U^- \times [\mathbb{A}^r / Z] \times U =: \mathcal{X}_0. \end{aligned}$$

Again, we call it the open cell of \mathcal{X} . The construction of the embedding \overline{H}_c and thus \mathcal{X} depend on a choice of a regular dominant weight λ but just as in the case of the wonderful compactification different choices of λ give isomorphic objects. Let us prove the following theorem

Theorem 2.9. *Let G be a semi simple group and \mathcal{X} as in definition 2.8. Then:*

- (a) \mathcal{X} is smooth and proper.
- (b) $\mathcal{X} - \mathcal{X}_0$ is of pure codimension 1 and we have an exact sequence

$$0 \rightarrow \mathbb{Z}^r \rightarrow \text{Pic}(\mathcal{X}) \rightarrow Z(G) \rightarrow 0$$

where the subgroup \mathbb{Z}^r is generated by the irreducible components of $\mathcal{X} - \mathcal{X}_0$.

- (c) The boundary $\mathcal{X} - G$ consists of r divisors D_1, \dots, D_r with simple normal crossings and the closure of the $G \times G$ -orbits are in bijective correspondence with subsets $I \subset [1, r]$ in such a way that to I we associate $\cap_{i \in I} D_i$.
- (d) Let u_1, \dots, u_r be generators of the rays of the Weyl chamber and M be the monoid they generate. Any G equivariant $\mathcal{X}' \rightarrow \mathcal{X}$ determines and is determined by a fan supported in the negative Weyl chamber whose lattice points lie in M .

Proof. The open cell is smooth and covers \mathcal{X} . Further \mathcal{X} is finite over the projective scheme \overline{G}_{ad} hence (a). For (b) we use that $Pic(\mathcal{X}) = Cl(X_c)$; this is shown in remark 3.4 of [14], where X_c is the coarse moduli space for \mathcal{X} . Working in X_c we see that as $X_{c,0} = U^- \times \overline{T}_c \times U$ is affine the complement is pure codimension 1 and we have an exact sequence

$$\mathbb{Z}^r \rightarrow Cl(X_c) \rightarrow Cl(X_{c,0}) \cong Cl(\overline{T}_c)$$

the map $\mathbb{Z}^r \rightarrow Cl(X_c)$ is injective because it can be identified with the image of $Pic(\overline{G}_{ad}) \cong \mathbb{Z}^r$ under pull back by $X_c \rightarrow \overline{G}_{ad}$ and one can choose ample generators $\in Pic(\overline{G}_{ad})$ that pulls back to an ample line bundles. Now $Cl(\overline{T}_c)$ can be computed algorithmically as

$$\mathbb{Z}^r \cong \Lambda_T \xrightarrow{m} \mathbb{Z}^r \cong Div_T(\overline{T}_c) \rightarrow Cl(\overline{T}_c)$$

where $Div_T(\overline{T}_c)$ is the subgroup of T -invariant Weil divisors and is generated by the rays $C^{(1)}$ of the fan C of \overline{T}_c , which is just the Weyl chamber. The map m a character $\chi \mapsto \sum_{D \in C^{(1)}} \langle \chi, D \rangle \cdot D$. In a suitable bases m is the Cartan matrix of \mathfrak{g} and consequently the cokernel of m is isomorphic to the weight lattice modulo the root lattice which is just $Z(G)$ when G is connected and simply connected. Altogether this gives (b).

For (c) it follows similarly as for \overline{G}_{ad} by reducing to the torus \overline{T}_0 . We can present \overline{T}_0 as $[\mathbb{A}^r/Z]$ where Z is a finite group so the result follows by observing that it holds for the atlas \mathbb{A}^r .

To prove (d) we first note that any equivariant map determines a map $\mathcal{X}'_0 \rightarrow \mathcal{X}_0$ and consequently determines a map of torics stacks $Y(\Sigma', \beta') \rightarrow [\mathbb{A}^r/Z]$. By [13, 3.4] this corresponds to a diagram

$$\Sigma' \longrightarrow \Sigma$$

$$\begin{array}{ccc} V_{T_{\Sigma'}} & \longrightarrow & V_{T_{\Sigma}} = \mathbb{Z}^r \\ \downarrow \beta' & & \downarrow \beta \\ V_{T'} & \longrightarrow & V_T \end{array}$$

Note that $\text{im } \beta = M$. This gives one direction. Conversely if we are given a fan whose lattice points lie in M then we can lift this data to a diagram as above; That is if F is our starting fan then there is a torus $T_{\Sigma'}$ and a map $V_{T_{\Sigma'}} \xrightarrow{\beta'} V_T$ and a fan Σ' such that $\beta'(\Sigma') = F$. Now apply proposition 2.5 to the fan $(\vec{0}, \Sigma')$ for the group $H' = G \times T_{\Sigma'}$. This admits a morphism to \overline{H}_c which descends to give a map to \mathcal{X} . \square

Remark 3. In general the generators of M don't generate all the lattice points of the Weyl chamber thus there are fans supported in the negative Weyl chamber which don't yield maps to \mathcal{X} ; one can simply take the fan that is the negative Weyl chamber with all its lattice points. This corresponds to the canonical embedding X_{can} of the group G and it does not admit a map to \mathcal{X} unless $\mathcal{X} = X_{can}$.

Remark 4. As noted in the introduction, this result has already been proved by Martens and Thaddeus in [23]. For parts (a),(d) see specifically [23, 4.2,6.4]. Part (c) seems implicit in the treatment given in [23] but part (b) seems to have been missed. Martens and Thaddeus give an alternative construction using the Vinberg construction mentioned in section 2. In [32], Vinberg shows there is morphism from $S^0 \rightarrow Cone(\overline{H}_c)$ where $Cone(\overline{H}_c)$ is the affine cone of \overline{H}_c and S^0 is explained in remark 2. This morphism descends and induces an equivariant morphism from Thaddeus and Martens' compactification $[S^0/(\mathbb{C}^\times)^n] \rightarrow \mathcal{X}$. It is an isomorphism over the open cell and consequently an isomorphism everywhere. Alternatively one can also see the equivalence between the Vinberg construction and given construction using universal torsors; this is explained in [5, 3.2.4].

Remark 5. In general $Pic(\mathcal{X})$ represents a nontrivial class in $Ext^1(Z(G), \mathbb{Z}^r) \cong \mathbb{Z}^r/Z(G)\mathbb{Z}^r$. The group $Z(G)$ is cyclic except in the case when G is of type B_n with n even, in which case $Z(G) = \mathbb{Z}/2 \times \mathbb{Z}/2$. In any case one can compute the class of $Pic(\mathcal{X})$ in $Ext^1(Z(G), \mathbb{Z}^r)$ by taking Weil divisors D associated to generators of $Z(G)$ then taking the appropriate multiple mD so that mD is Cartier. Then mD is

represented by a class in \mathbb{Z}^r and the class of mD under the composition $\mathbb{Z}^r \rightarrow \text{Ext}^1(Z(G), \mathbb{Z}^r)$ gives the class of $\text{Pic}(\mathcal{X})$.

Remark 6. Of course this stacky construction extends to a general reductive group over \mathbb{C} . But one thing that changes is that for a general reductive group the Weyl chamber is not a strongly convex rational polyhedral fan. That is, there is not a toric variety associated to the Weyl chamber; this happens for GL_n . Thus to get embedding of such groups we must further subdivide the Weyl chamber.

Kausz does this for GL_N as follows. Let $\phi: GL_N \rightarrow PGL_{N+1}$ be given by composing $g \mapsto \begin{pmatrix} g & 0 \\ 0 & (\det g)^{-1} \end{pmatrix}$ with the natural map $SL_{N+1} \rightarrow PGL_{N+1}$. This induces a map T_{GL_N} to $T_{PGL_{N+1}}$ and an induced map on co-character $\phi_*: V_T^{GL_N} \rightarrow V_T^{PGL_{N+1}}$. Let Σ be the fan associated to the Weyl chamber decomposition for PGL_{N+1} and Σ' the pull back of Σ under the map induced by ϕ_* . Then Kausz compactification is the one associated to Σ' via proposition 2.5. Certainly there are other possibilities.

3. EXTENSION TO LOOP GROUPS

We now extend the results of section 2.2 to loop groups. In this section we recall some standard results on loop groups and their representations.

As mentioned in the introduction there is both the algebraic loop group and the smooth loop group. In this section we focus on the algebraic loop group. To a large extend the statements we prove also hold for the smooth loop group. We remark along the way what modifications, if any, are necessary to get a statement for the smooth loop group. When we apply these results to studying the moduli stack of bundles on curves we will be more explicit about the distinction between LG and $L^{sm}G$.

Let G be a connected simple algebraic group over \mathbb{C} with $\pi_1(G) = 1$. The loop group LG is the functor from \mathbb{C} -algebras to groups given by

$$R \mapsto LG(R) := G(R((z))) := G(\text{Spec } R((z)))$$

The functor LG is represented by an ind-scheme of infinite type; see [9]. Much like a reductive group has standard subgroups (T, B^\pm, U^\pm, \dots etc) the loop group has loop analogues of these subgroups and others as well. Below are the groups we most frequently utilize

$$\begin{array}{ll} L^+G(R) = G(R[[z]]) & \text{positive loop group} \\ L^-G(R) = G(R[[z^{-1}]]) & \text{negative loop group} \\ \mathcal{B}^\pm = \{\gamma \in L^\pm G \mid \gamma \in B^\pm \text{ mod } \pm z\} & \text{loop analogue of } B^\pm \\ \mathcal{U}^\pm = \{\gamma \in L^\pm G \mid \gamma \in U^\pm \text{ mod } \pm z\} & \text{loop analogue of } U^\pm \\ L^\times G = \mathbb{C}^\times \ltimes LG & \\ \tilde{L}G & \text{Central Extension of } LG \\ G^{aff} = \mathbb{C}^\times \ltimes \tilde{L}G & \text{affine Kac-Moody group of } G \end{array}$$

In the sequel when referring to the positive versions we simply write \mathcal{B} and \mathcal{U} . The action of $u \in \mathbb{C}^\times$ on LG that defines the semidirect product in $L^\times G$ is

$$\left\{ \text{Spec } R((z)) \rightarrow G \right\} \xrightarrow{\circ u} \left\{ \text{Spec } R((z)) \xrightarrow{z \mapsto uz} \text{Spec } R((z)) \rightarrow G \right\}$$

The notation $L^\times G$ is not standard. When we need to differentiate between the \mathbb{C}^\times in $L^\times G$ and the \mathbb{C}^\times defining the central extension we will denote the latter by \mathbb{C}_c^\times :

$$0 \rightarrow \mathbb{C}_c^\times \rightarrow \tilde{L}G \rightarrow LG \rightarrow 0$$

This central extension will play an important role when we discuss line bundles on the embedding we define. However we will not discuss the full group structure of G^{aff} as it does not enter into our construction; for more details see e.g. [20, chap. 13]. When H is a subgroup of LG we write \tilde{H} for the restriction of the central extension to H . Similarly, for any subgroup $H \subset LG$ we denote by H^\times the semi-direct product $\mathbb{C}^\times \ltimes H$. We also do this for subgroups of G even though the conjugation action on G is trivial; for example we write the maximal torus of $L^\times G$ as $T^\times = \mathbb{C}^\times \times T$.

We make extensive use of the refined Birkoff factorization [20, pg.142]

$$\begin{aligned} LG &= \bigsqcup_{w \in W^{aff}} \mathcal{U}^- \cdot w \cdot \mathcal{B} \\ L^\times G &= \bigsqcup_{w \in W^{aff}} \mathcal{U}^- \cdot w \cdot \mathcal{B}^\times \\ G^{aff} &= \bigsqcup_{w \in W^{aff}} \mathcal{U}^- \cdot w \cdot \mathcal{B}^{aff} \end{aligned} \tag{7}$$

where $W^{aff} = W \ltimes V_T$ is the affine Weyl group; here W is the Weyl group of G .

All of this notation has a counterpart for the smooth loop group $L^{sm}G = C^\infty(S^1, G(\mathbb{C}))$; it is simply a matter of specifying what are positive and negative loops. For example if we identify S^1 with $\{|z| = 1\} \subset \mathbb{C}$ then L^+G = boundary values of holomorphic maps $\{|z| < 1\} \rightarrow G$ and L^-G is the subgroup of boundary values of holomorphic maps $\{|z| > 1\} \rightarrow G$. A reference for the smooth version of (7) is [26, 8.7.3a].

3.1. Representation Theory of G^{aff} . Let r be the rank of G and $\alpha_1, \dots, \alpha_r$ be the simple roots of G . The maximal torus of G^{aff} is $T^\times \times \mathbb{C}_c$ and characters are denoted as $(n, \mu, l) \in \mathbb{Z} \oplus \Lambda_T \oplus \mathbb{Z}$. The simple affine roots of G^{aff} are $(0, \alpha_1, 0), \dots, (0, \alpha_r, 0), \alpha_0 = (1, -\theta, 0)$ where θ is the longest root of G . By abuse of notation we denote $(0, \alpha_i, 0)$ simply by α_i so the roots of G^{aff} are $\alpha_0, \dots, \alpha_r$. Let $\omega_r, \dots, \omega_1$ be the fundamental weights of G . The fundamental weights of G^{aff} are $\tilde{\omega}_0 = (0, 0, 1), \tilde{\omega}_1 = (0, \omega_1, 1), \dots, \tilde{\omega}_r = (0, \omega_r, 1)$. We also have the dual fundamental weights $-\tilde{\omega}_i$; they generate the anti-dominant weights. In the smooth loop group setting it is convention to consider lowest weight representation which are better known as positive energy representations. Because (1) uses both a representation and its dual we make use of both highest weight representations and positive energy representations.

The Lie algebra of $G^{aff}(\mathbb{C})$ is, as a vector space,

$$\mathfrak{g}^{aff} = \mathbb{C} \oplus \mathfrak{g}[z^{\pm}] \oplus \mathbb{C}_c$$

Thus the roots of G^{aff} are $(k, \alpha, 0)$ where α is a root of G . For $k \neq 0$ $\alpha = 0$ is allowed so in general the weight spaces are not 1 dimensional as $\dim \mathfrak{g}_{(k \neq 0, 0, 0)} = \dim \mathfrak{t}$; these are the imaginary roots of $\mathbb{C} \oplus L\mathfrak{g}$. The roots are linear forms on $\mathbb{R} \oplus \mathfrak{t}$. Traditionally they are identified with affine linear forms on \mathfrak{t} by identifying the Lie algebra with $1 \oplus \mathfrak{t}$.

For $\alpha \neq 0$ we can define affine hyperplane in \mathfrak{t} via

$$H_{k, \alpha} = \{\zeta \in \mathfrak{t} | \alpha(\zeta) = -k\}$$

The complement of all the $H_{k, \alpha}$ is known as the *Weyl alcove decomposition* of \mathfrak{t}

Proposition 3.1. *Let $\tilde{\lambda} = (0, \lambda, h)$ be a regular dominant weight. There exists a representation $V = V(\tilde{\lambda})$ of G^{aff} with the following properties*

- (a) *If $\tilde{\mu}$ is any other weight of L then $\tilde{\lambda} - \tilde{\mu}$ is a sum of positive roots.*
- (b) *$\tilde{\lambda} - \alpha_i$ is a weight of V for all i .*
- (c) *The stabilizer of the weight space $V_{\tilde{\lambda}}$ in $\mathbb{P}V$ is \mathcal{B}^{aff} .*
- (d) *The morphism $G^{aff}/\mathcal{B}^{aff} = LG/\mathcal{B} \rightarrow \mathbb{P}(L)$ given by $\gamma\mathcal{B} \mapsto \gamma L_{\tilde{\lambda}}$ is injective and gives LG/\mathcal{B} the structure of a projective ind scheme; in particular LG/\mathcal{B} is closed in $\mathbb{P}(L)$.*
- (e) *The action of G^{aff} on $\mathbb{P}(L)$ factors through a faithful action of $G^{aff}/Z(G^{aff}) = L^\times G/Z(G)$.*

Proof. The existence claim is contained in [20, 13.2.8].

- (a) [20, 1.3.22].
- (b) This follows from [20, 1.3.22] and the representation theory of SL_2 . Namely, for each simple root, consider the reflection $\tilde{s}_i(\tilde{\lambda})$. It is of the form $\tilde{\lambda} - n\alpha_i$ for $n \geq 1$ and all the weights $\tilde{\lambda} - m\alpha$ for $0 \leq m < n$ are weights of the representation.
- (c) [20, 7.1.2]
- (d) [20, Ch.7.1]

(e) [20, 13.2.8]. More specifically we have the following commutative diagram

$$\begin{array}{ccc} G^{aff} & \longrightarrow & GL(V) \\ \downarrow \psi & & \downarrow \\ L^\times G/Z(G) & \longrightarrow & PGL(V) \end{array}$$

where ψ is a surjective group homomorphism and $\ker \psi = Z(G^{aff})$.

□

Remark 7. The statements (a) - (c) and (f) for $L^{sm}G$ are proved in [26, ch.8,9,11]. The first half of (d) also holds [26, 8.7.6] but there is no ind-statement to make about the flag varieties of the smooth loop group; they are infinite dimensional complex projective algebraic varieties.

Remark 8. The dual U^* of a highest weight representation U is $\prod_\mu \text{hom}(U_\mu, \mathbb{C})$; it is the completion of the restricted dual $U_{res}^* := \oplus \text{hom}(U_\mu, \mathbb{C})$. The smooth loop group acts on U^* but in fact in this case there is a way to give this representation a unitary structure and complete U_{res}^* with respect to the associated norm to produce an intermediate vector space $U_{res}^* \subset U^{pos} \subset U^*$. The representation U^{pos} is called a positive energy representation of $L^{sm}K$ or $L^{sm}G$. We primarily work with U and U^* .

The defining characteristic of all these representations are that the \mathbb{C}^\times -action (in the semi direct product) is bounded either above or below and in addition all the weight spaces are finite dimensional. An example is discussed in more detail in section 4.

Statements (a), (b), (c), (e) still hold for LG acting on U^* . Again, statement (d) is different because LG/\mathcal{B}^- does not have the structure of a finite type ind variety. In fact LG/\mathcal{B}^- is a scheme of infinite type. It is a closed subscheme of $\mathbb{P}U^*$ by lemma 3.2 and it has a cover by affine schemes isomorphic to \mathcal{U} ; this can be seen from the Birkhoff decomposition (7). Statement (d) would not be true for the formal loop group if we worked with the restricted dual $U_{res}^* := \oplus_\mu \text{hom}(U_\mu, \mathbb{C})$ because for $v \in U_{res}^*$ and $\gamma \in G((z))$ we in general have $\gamma.v \in U^* - U_{res}^*$. If we work with the polynomial loop group $L_{poly}G(R) = G(R[t^\pm])$ the proposition 3.1 holds equally well for both U, U_{res}^* .

Now we prove the appropriate analogue of lemma 2.4. Note that in this infinite dimensional setup closure is more subtle. For example consider the map $\mathbb{C}^\times \rightarrow U = \prod_{n \in \mathbb{N}} \mathbb{C}$ given by $t \mapsto (1, t, t^{-1}, t^2, t^{-2}, \dots)$. Then the limit as $t \mapsto 0$ does not exist in $\mathbb{P}U$. In light of this we understand $Y \subset \mathbb{P}U$ to be closed if the inclusion $Y \rightarrow \mathbb{P}U$ satisfies the existence part of the valuative criterion for properness. Colloquially Y is closed if any limit point of Y that exists in $\mathbb{P}U$ belongs to Y .

Lemma 3.2. *Let G be a Kac-Moody group and let V a highest weight representation of G . The orbit of a highest weight is the only closed orbit in $\mathbb{P}V$. The orbit of a lowest weight vector is the only closed orbit in $\mathbb{P}V_{res}^*, \mathbb{P}V^*$.*

Proof. First some notation. For each root α we get a root subgroup $\mathbb{G}_\alpha \cong U_\alpha \subset G$. By $U_\alpha(b)$ we denote the group element $\exp(b \cdot X_\alpha)$ where $b \in \mathbb{C}$. The representation V is in particular an integrable representation of \mathfrak{g}^{aff} thus X_α , for α positive, acts locally nilpotently on V . Thus for any v the vector $U_\alpha(b).v$ is a polynomial function of b ; that is it has bounded exponents of b so the limit $b \mapsto 0$ exists in $\mathbb{P}U$ as well as the limit $b \mapsto \infty$; we write the latter as the limit $b \mapsto 0$ of $U_\alpha(1/b).v$.

Suppose the orbit of v is closed in $\mathbb{P}V$. Then for any 1-parameter subgroup $\mathbb{G}_a \rightarrow G$ we have the closure $\overline{\mathbb{G}_a.v}$ is also in the orbit of v . Write v as a sum of weight vectors

$$v = \sum_i v_{\mu_i} \tag{8}$$

Let λ be the highest weight of V . Thinking of V as a quotient of a Verma module, we can think of each weight vector as $Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_m}} \otimes 1_\lambda$ where the α_{i_j} are not necessarily distinct; here we write Y_α for $X_{-\alpha}$.

Now look at the limit of $b \mapsto 0$ of $U_{\alpha_1}(1/b).v$. Call the resulting point $v_1 \in \mathbb{P}V$. Note v_1 is necessarily a sum of weight vectors $Y_{\alpha_{i_1}} \cdots Y_{\alpha_{i_m}} \otimes 1_\lambda$ where no α_{i_j} has α_1 in its support; that is no $\alpha_{i_j} = \alpha_1 + \alpha_{i_j'}$.

For if this was not the case then the Lie algebra action of $X_{\alpha_1}.v_1 \neq 0$ so U_{α_1} acts non trivially on v_1 . However,

$$U_{\alpha_1}(b).v_1 = U_{\alpha_1}(b). \lim_{a \rightarrow 0} U_{\alpha_1}(1/a).v = \lim_{a \rightarrow 0} U_{\alpha_1}(b/a).v = \lim_{a \rightarrow 0} U_{\alpha_1}(1/a).v = v_1.$$

Repeating this process with α_2 we get a vector $v_{1,2}$ which is a sum of weight vectors with no α_{i_j} having α_2 in its support. Also note that the action of $U_{\alpha_2}(b)$ preserves the property that no α_{i_j} has α_1 in its support. Continuing we get a vector $v_{1,2,\dots,r}$ which must necessarily be the class of the highest weight vector in $\mathbb{P}V$.

The same argument goes through for U^* and U_{res}^* . All that changes for U^* is the sum (8) is now potentially infinite but we can correct this by replacing v with $\lim_{s \rightarrow 0} \eta(s).v$ for a generic $\eta \in V_{T^\times}$. In this case η acts nontrivially on the v_μ and $v' = \lim_{s \rightarrow 0} \eta(s).v$ is supported in the weight spaces V_μ such that μ minimizes the function $\chi \mapsto \langle \chi, \eta \rangle$. This function is a quadratic function on a lattice so there are only finitely many μ that minimize it; see section 4 for more details. Therefor we have reduced to case that the sum in (8) is finite.

We have shown if there is a closed orbit it must be that of a highest weight. Let us show this is indeed closed. Let Y be the closure in $\mathbb{P}U$ of the orbit of a highest weight vector v . Suppose $[w] \in Y - G.[v]$; as $Y - G.[v]$ is G -stable and closed. On the other hand, using the method above, we can find a sequence of vectors $w = w_1, \dots, w_r = v$ such that

$$w_{i+1} \in \overline{G.w_i} \subset \overline{G.v} - G.v.$$

In particular $v = w_r \in \overline{G.v} - G.v$ which is clearly a contradiction. \square

Remark 9. This result relies only on the representation theory of LG and it holds for $L^{sm}G$ acting on U^{pos} .

3.2. The Wonderful Construction. Let G be a connected, simply connected, simple algebraic group over \mathbb{C} or rank r . Fix now a regular dominant weight $\tilde{\lambda}$ for the affine form $G^{aff} = \widehat{L^\times}G$ of G and a representation $V = V(\tilde{\lambda})$. We have an action of $G^{aff} \times G^{aff}$ on $V \hat{\otimes} V^* := \prod_{\mu, \nu} \text{hom}(V_\mu, V_\nu)$ given by $(\gamma_1, \gamma_2).\phi = \gamma_1 \circ \phi \circ \gamma_2^{-1}$. Just as in section 2 we define

$$X^{aff} := \overline{G^{aff} \times G^{aff}.[id]} \subset \mathbb{P}[\tilde{\lambda}] := \mathbb{P}V \hat{\otimes} V^* \quad (9)$$

It follows from (e) of proposition 3.1 that the stabilizer of the identity is $Z(G^{aff}) \cdot \Delta(G^{aff})$ and so X contains $G^{aff}/Z(G^{aff}) = L^\times G/Z(G)$ as a dense open subset. X^{aff} is an ind scheme but we postpone showing this until the next section.

The goal now is to prove 3.7 which is the analogue of 2.1 for the loop group. As in section 2 we define the spaces

$$\mathbb{P}_0 = \{v_{\tilde{\lambda}} \otimes v_{\tilde{\lambda}}^* \neq 0\} \quad X_0^{aff} = X^{aff} \cap \mathbb{P}_0 \quad \overline{T_{ad,0}^\times} = \overline{T_{ad}^\times} \cap \mathbb{P}_0$$

Let us note that lemma 3.2 give us

$$X^{aff} = G^{aff} \times G^{aff}.X_0^{aff}; \quad (10)$$

see (4) and the explanation below it.

Remark 10. The following results (3.3 to 3.6) are proved for the algebraic loop group but they make sense are true for the smooth loop group. This is because they depend only on the geometry of T, G , the Birkhoff factorization (7) and the Lie algebra action of the root spaces on a representation which are the same for both groups.

The same proof as in lemma 2.7 gives

Proposition 3.3. $\overline{T_{ad,0}^\times} \cong \mathbb{C}[t^{-\alpha_0}, \dots, t^{-\alpha_l}] \cong \mathbb{A}^{l+1}$. In particular $\overline{T_{ad}^\times}$ is smooth and its fan is given by the negative Weyl alcove; the fan $\overline{T_{ad}^\times}$ is given by the Weyl alcove decomposition of $1 \oplus V_{T_{ad}} \otimes \mathbb{R}$.

Proof. We just need the second statement; this follows because the negative Weyl alcove is a fundamental domain for the action of the affine Weyl group on $1 \oplus V_{T_{ad}} \otimes \mathbb{R}$. \square

We focus now on analyzing the $\mathcal{U} \times \mathcal{U}^-$ action on X_0 . Let $v = v_{\bar{\lambda}} \in V$ and $v^* \in V^*$ the dual vector. Define $\mathbb{P}_v := \{v \neq 0\} \subset \mathbb{P}V$ and $\mathbb{P}_{v^*} := \{v^* \neq 0\} \subset \mathbb{P}V^*$. The proofs of lemma 3.4 and proposition 3.5 are adapted from [6, 6.1.7]; the extension for the loop group essentially requires replacing the Bruhat decomposition of a reductive group with the Birkhoff factorization of LG .

Lemma 3.4. $LG.v \cap \mathbb{P}_v = \mathcal{U}^-.v$ and $LG.v^* \cap \mathbb{P}_{v^*} = \mathcal{U}.v^*$. In particular X_0 is $\mathcal{U} \times \mathcal{U}^-$ stable.

Proof. Let $\pi: V \rightarrow \mathbb{C} \cdot v$ be the projection. Let $f_{\pi}(g) = \pi(g.v)$. We have $f_{\pi}(g) = 0$ if and only if $g.v \cap \mathbb{P}_v = \emptyset$. By the Birkhoff decomposition any $g = u_{-}.w.b$ with $u_{-} \in \mathcal{U}^-$, $w \in W^{aff}$ and $b \in \mathcal{B}$. It follows that $f_{\pi}(g) = 0$ if and only if $f_{\pi}(u_{-}.w) = 0$. If $v' = v_{\mu} + \sum_{\nu < \mu} v_{\nu}$ and $u \in \mathcal{U}^-$ then $u.v_{\mu}$ still has nonzero projection to v_{μ} and all other weights are still less than μ . Consequently $f_{\pi}(u_{-}.w) \neq 0$ if and only if $f_{\pi}(w) \neq 0$ if and only if $w = 1$. Therefore

$$LG.v \cap \mathbb{P}_v = \mathcal{U}^-.v \cap \mathbb{P}_v = \mathcal{U}^-.v \cong \mathcal{U}^-.$$

The second equality follows because the stabilizer of $[v]$ in $\mathbb{P}V$ is \mathcal{B} . The same argument works with LG acting on V^* . \square

Remark 11. Of course we can also replace LG with $L^{\times}G$ or G^{aff} .

Proposition 3.5. *There is an $\mathcal{U} \times \mathcal{U}^-$ equivariant isomorphism*

$$\begin{aligned} \mathcal{U}^- \times \mathcal{U} \times \overline{T_{ad,0}^{\times}} &\xrightarrow{a} X_0^{aff} \\ (l, u, t) &\mapsto l \cdot t \cdot u \end{aligned}$$

Proof. First note that the restriction to $\mathcal{U}^- \times \mathcal{U} \times T_{ad}^{\times}$ is just the multiplication map and this is known to be open by the Birkhoff decomposition; consequently the morphism is birational. The next step is to construct a $\mathcal{U} \times \mathcal{U}^-$ -equivariant map $X_0^{aff} \rightarrow \mathcal{U} \times \mathcal{U}^-$.

Let $\phi \in X_0$ then thinking of ϕ as an endomorphism defined away from $\ker \phi$ we see that $\phi(v)$ is defined and is in \mathbb{P}_v . On the other hand $\phi(v)$ is in the closure of the orbit of the highest weight, but this orbit is closed by lemma 3.2 hence $\phi(v) \in L^{\times}G.v$. So by the previous lemma $\phi(v) = l.v$ for a unique l . We get a map $X_0^{aff} \rightarrow \mathcal{U}^-$ via $\phi \mapsto l$. Similarly we can get a map $X_0^{aff} \rightarrow \mathcal{U}$. Altogether we have a map

$$X_0^{aff} \xrightarrow{b} \mathcal{U}^- \times \mathcal{U}$$

The composition $\mathcal{U}^- \times \mathcal{U} \times T_{ad}^{\times} \rightarrow X_0^{aff} \rightarrow \mathcal{U}^- \times \mathcal{U}$ is given by $(l, t, u) \mapsto (l, u)$.

To finish we show $\mathcal{U}^- \times \mathcal{U} \times b^{-1}(1, 1) \xrightarrow{a} X_0^{aff}$ is bijective and $b^{-1}(1, 1) = \overline{T_{ad,0}^{\times}}$. For injectivity note that as $b^{-1}(1, 1)$ is a subset of X_0^{aff} it suffices to show that if $a(l, t, u) = a(l', t', u') = x$ then $u = u', l = l'$. This follows

$$(l, u) = b \circ a(l, t, u) = b(x) = b \circ a(l', t', u') = (l', u').$$

Now surjectivity. Let $\phi \in X_0^{aff}$ and $(l, u) = b(\phi)$. Then $t := (l^{-1}, u^{-1}).\phi \in b^{-1}(1, 1)$, hence (l, t, u) does the job.

It remains to show $b^{-1}(1, 1) = \overline{T_{ad,0}^{\times}}$. Clearly we have \supset as b^{-1} is closed and contains T_{ad}^{\times} and as a is birational it follows that they have the same dimension. Now $\pi_0(LG) = \pi_1(G) = 0$. Further, the map $G \rightarrow G/Z(G) =: G_{ad}$ induces a map $LG \rightarrow LG_{ad}$; the image is the connected component of the identity, in particular it is irreducible. It follows that X^{aff} and X_0^{aff} are irreducible hence so is $X_0^{aff}/\mathcal{U} \times \mathcal{U}^- \cong b^{-1}(1, 1)$. Thus it must equal $\overline{T_{ad,0}^{\times}}$. \square

The next result is known in the finite dimensional case; for example a similar result appears in [32]. The boundary of X^{aff} is $L^{\times}G \times L^{\times}G$ stable and so breaks up into a disjoint union of $L^{\times}G \times L^{\times}G$ orbits. Also these $L^{\times}G \times L^{\times}G$ orbits all intersect X_0^{aff} . The standard idempotents $e_I = \sum_{i \in I} e_i \in \overline{T_{ad,0}^{\times}}$ for $I \subset \{1, \dots, r+1\}$ lie in distinct $L^{\times}G \times L^{\times}G$ orbits and as these are representatives for the $T^{\times} \times T^{\times}$ orbits in $\overline{T_{ad,0}^{\times}}$ it follows that $X^{aff} - L^{\times}G/Z(G) = \bigsqcup_{I \subset [1, r]} L^{\times}G \times L^{\times}G / \text{Stab}(e_I)$.

To compute $\text{Stab}(e_I)$ we need to recall some facts about parabolic subgroups. For each subset of $I \subset \{\alpha_1, \dots, \alpha_r\}$ of the simple roots we have a standard parabolic Lie subalgebra. Specifically, if Δ is the set of roots of \mathfrak{g} , then we have a parabolic Lie algebra $\mathfrak{p}_I \subset \mathfrak{g}$ generated by $\oplus_{i \in I} \mathbb{Z}\alpha_i \cap \Delta$ and Δ^+ and this integrates to a standard parabolic subgroup P_I .

Proposition 3.6. *For $J \subset [1, r+1]$ let P_J^\pm be the associated opposite parabolics with standard Levi decomposition $P_J = L_J \cdot U_J^\pm$. Set $S(J) = \{(g_1, g_2) \in P_J \times P_J^- : g_1|_{L_J} = g_2|_{L_J}\} = \Delta(L_J) \ltimes (U_J^- \times U_J)$. Also define a group $\ker(J) = \cap_{j \notin J} \ker \alpha_j$ and $T(J) := \ker(J) \times \ker(J)$. Then*

$$\begin{aligned} \text{Stab}(e_J) &= T(J) \cdot S(J) \\ X - L^\times G / Z(G) &= \bigsqcup_{J \subset [1, l]} L^\times G \times L^\times G / T(J) \cdot S(J) \end{aligned}$$

Proof. That $T(J)$ is in the stabilizer follows from the description of \overline{T}_0 given in the proof lemma 2.7. So let us focus on the group $S(J)$.

The group $L^\times G \times L^\times G$ and in particular the standard parabolic subgroups are generated by the root subgroups $U(\alpha) \cong \mathbb{G}_a$ which it contains. So to check $S(J)$ is in the stabilizer it suffices to check it for the root subgroups it contains. These break up into two cases. Roots subgroups of the form $(U(\alpha), 1)$ or $(1, U(\alpha))$ and those of the form $\Delta(U(\alpha))$. In simplified context one can consider the stabilizer of a matrix

$$\begin{bmatrix} I_n & 0 \\ 0 & 0 \end{bmatrix}$$

in $GL(V) \times GL(V)$. One easily checks the stabilizer are pairs of block matrices and the two cases above correspond to elements in the stabilizer

$$\begin{bmatrix} I & B \\ 0 & I \end{bmatrix}, \begin{bmatrix} I & 0 \\ C & I \end{bmatrix} \\ \Delta\left(\begin{bmatrix} A & 0 \\ 0 & D \end{bmatrix}\right)$$

First let us check $(U(\alpha), 1)$ is in $\text{Stab}(e_J)$. As $\alpha \notin \text{Lie}(L_J)$ we have that $\alpha = \alpha_i + \alpha'$ for some $i \notin J$. It suffices to check that $X_{\alpha \cdot e_J} = 0$. Recall e_J is an idempotent of $\text{End}(V(\tilde{\lambda}))$ and we can express $e_J = \sum_j e_j \otimes e_j^*$ where j ranges over some subset of the weights of the representation. Therefore to show $X_{\alpha \cdot e_J} = 0$ it suffices to show $X_{\alpha \cdot v_J} = 0$ where $v_J = \sum_j e_j$. Assuming the contrary, $X_{\alpha \cdot v_J} \neq 0$ implies $X_{\alpha \cdot e_j} \neq 0$ for some j . The weight j has the property that $\lambda - j \in \sum_{i \in J} n_i \alpha_i$ with $n_i \geq 0$. But if $e_\mu := X_{\alpha \cdot e_j}$ is not zero then it is a weight vector of weight $\mu = \alpha + j$. But then $\lambda - \mu$ fails to be a sum of positive roots, contradiction.

Now if $\alpha = \sum_{j \in J} n_j \alpha_j$ then X_α, Y_α both preserve the vector space spanned by e_j where again we write $v_J = \sum_j e_j$. Thus we can restrict to this subrepresentation and we are reduced to computing the stabilizer of the identity in a representation. In particular we conclude that all $(U(\alpha), U(\alpha))$ are in the stabilizer.

This shows that $T(J)S(J)$ is contained in the stabilizer, and for codimensional reasons there can't be a higher dimensional group that stabilizes e_J . As $L^\times G$ is connected, there can't be other components. \square

Theorem 3.7. *Let $X^{aff} = \overline{G^{aff}/Z(G^{aff})}$ be as in (9). Then*

- (a) X^{aff} is independent of λ .
- (b) $X^{aff} - X_0^{aff}$ is of pure codimension 1. It is a union of $r+1$ divisors that generate the Picard group.
- (c) The boundary $X^{aff} - L^\times G / Z(G)$ consists of $r+1$ normal crossing divisors D_1, \dots, D_r and the closure of the $G^{aff} \times G^{aff}$ -orbits are in bijective correspondence with subsets $I \subset [1, r+1]$ in such a way that to I we associate $\cap_{i \in I} D_i$.
- (d) Any G equivariant $X' \rightarrow X^{aff}$ determine and is determined by a weyl equivariant morphism of toric varieties $\overline{T}' \rightarrow \overline{T}_{ad,0}^\times$.

Proof. The proof of (a) is the same as 2.1(a). For (b) we use [25, 2.4] which states that the complement of a dense open subset in any scheme is of pure codimension one. To apply it we utilize the ind scheme structure on X^{aff} given in proposition 4.2. Namely we have ind-scheme structures

$$X^{aff} = \cup_n X_n^{aff} \quad X_0^{aff} = \cup_n X_{n,0}^{aff}$$

and for every n $X_{n,0}^{aff}$ is a dense affine scheme in X_n^{aff} hence $X_n^{aff} - X_{n,0}^{aff}$ gives a sequence of compatible divisors and shows that $X^{aff} - X_0^{aff}$ itself is pure codimension 1.

Alternatively, any point $p \in X^{aff} - L^\times G/Z(G)$ can be expressed as $g_1 \cdot \eta(0) \cdot g_2$ with $g_i \in L^\times G$ and η is a co-character in the Weyl alcove. For $s \neq 0$ we have $g_1 \cdot \eta(s) \cdot g_2$ lies in the same Birkhoff coset $\mathcal{U}^- w \mathcal{B}$. Clearly $p \in \overline{\mathcal{U}^- w \mathcal{B}}$. Now $X^{aff} - X_0^{aff}$ is the closure of $L^\times G/Z(G) - \mathcal{U}^- T_{ad}^\times \mathcal{U}$. From [20, 7.1.21] and using the Birkhoff factorization we conclude

$$L^\times G/Z(G) - \mathcal{U}^- T_{ad}^\times \mathcal{U} = \bigcup_{i=0}^r D_i$$

where i is a divisor associated to the simple reflection $s_i \in W^{aff}$:

$$D_i = \overline{\mathcal{U}^- s_i \mathcal{B}} = \bigsqcup_{v \geq s_i} \mathcal{U}^- v \mathcal{B}$$

A precise description of order \geq is in [20, 1.3.15]; all we need is that for $w = s_i$ we have $v \geq s_i$ if and only if s_i appears in a reduced expression for v . It follows that

$$X^{aff} - X_0^{aff} = \bigcup_{i=0}^r \overline{D_i} \quad (11)$$

is of pure codimension 1. That the $\overline{D_i}$ are Cartier and generate the Picard group is proved in the next subsection in proposition 3.11.

For (c) we note that any $G^{aff} \times G^{aff}$ -orbit intersects $\overline{T_{ad,0}^\times}$ along a unique $T_{ad}^\times \times T_{ad}^\times$ -orbit. This follows from the description of the stabilizers given in proposition 3.6 and the fact that the standard parabolic subgroups in a Tits system are not conjugate [16, 30.1]. Therefore we can reduce the statement to the torus for which it is obvious.

The same argument for 2.1(d) works for (d) with one minor adjustment. In the finite dimensional case we utilized regular dominant representation with no further qualification. In the loop group setting, when working with multiple regular dominant representations, we must be sure that they are all of the same level. That is the central \mathbb{C}^\times acts by the same character on all the representations. Representations $(0, \lambda, l)$ of level l are characterized by $\lambda(\theta^\vee) \leq l$; here θ^\vee is the co-root associated to the longest root θ of G . It follows for any finite set $\lambda_1, \dots, \lambda_m$ of regular dominant weights there is a fixed l such that $(0, \lambda_i, l)$ are all regular dominant. \square

Remark 12. Proofs for (a),(c),(d) hold for the smooth loop group $L^{sm}G$ without further comment. Statement (b) is true for the loop group but as $L^{sm}G$ is not an ind object the argument is different. Equation (11) and the argument before it holds for $L^{sm}G$ and we can finish the proof of statement (b) without proposition 3.11. Comparing with the proof of part (c) of 2.1 it is just necessary to show $Pic(X_0^{aff}) = 0$. Given 3.5 and the fact that for the smooth loop group we have an abstract isomorphism $\mathcal{U} \cong \mathcal{U}^-$, the previous assertion reduces to $Pic(\mathcal{U}) = H^1(\mathcal{U}, \mathcal{O}^*) = 0$. This follows from a standard argument that asserts line bundles on \mathcal{U} are topological together with the result [26, 8.7.4 (ii)] which asserts that \mathcal{U} is contractible. To show line bundles are topological let $\mathcal{O}_{cts}, \mathcal{O}_{cts}^*$ represent the sheaf of continuous and non vanishing continuous functions on \mathcal{U} . Then we have a commutative diagram (suppressing the mention of \mathcal{U})

$$\begin{array}{ccccccc} H^1(\mathcal{O}) & \longrightarrow & H^1(\mathcal{O}^*) & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}) \\ \downarrow & & \downarrow & & \parallel & & \downarrow \\ H^1(\mathcal{O}_{cts}) & \longrightarrow & H^1(\mathcal{O}_{cts}^*) & \longrightarrow & H^2(\mathbb{Z}) & \longrightarrow & H^2(\mathcal{O}_{cts}) \end{array}$$

The vanishing of $H^i(\mathcal{O}_{cts})$ for $i \geq 1$ follows because \mathcal{U} is paracompact and the vanishing of $H^i(\mathcal{O})$ for $i \geq 1$ is proved in [22, 1.1]. Altogether we have $H^1(\mathcal{O}^*) = H^1(\mathcal{O}_{cts}^*)$ as required.

Let us now address the issue of completeness. First let \mathcal{F} be the fan of the closure of T^\times in X^{aff} and let $\eta \in F$ be a co-character. Certainly any map $\mathbb{C}((s)) \xrightarrow{m} L^\times G/Z(G)$ which factors as

$$m = h_1 \cdot \eta(s) \cdot h_2 \quad h_i \in L^\times G/Z(G)[[s]] \quad (12)$$

extends to a map $\mathbb{C}[[s]] \rightarrow X^{aff}$. In the finite dimensional case all maps $\mathbb{C}((s)) \rightarrow G$ admit such a factorization; that is we have the Iwahori decomposition: $G((s)) = G[[s]]T((s))G[[s]]$. But this does not hold for the loop group. One way to see this is by choosing a faithful irreducible representation $G \subset SL(W)$ and then one can identify elements of $LG((s))$ with expressions

$$\gamma(z) = \sum_{j=-k}^{\infty} a_j z^j \quad a_j \in \text{End}(W) \otimes \mathbb{C}((s)) \cong \mathbb{C}((s))^{(\dim W)^2}. \quad (13)$$

In other words we can identify them with matrices with entries in the field $R = \mathbb{C}((s))((z))$ of Laurent series in z with coefficients in $\mathbb{C}((s))$. Then consider a root subgroup $\mathbb{G}_\alpha \cong U_\alpha \subset G$. It follows that an arbitrary element of $\mathbb{G}_\alpha(R) = R$ is an element of $LG((s))$. But such an element gives an expression (13) which in general has unbounded poles of s whereas any expression coming from an element of form (12) has bounded poles of s .

So we do not have a completeness statement stronger than the one that holds for expression (12). Whether there are elements of $L^\times G/Z(G)((s))$ that are not of the form (12) that nevertheless extend to a map $\mathbb{C}[[s]] \rightarrow X^{aff}$ is a subtle question. For the smooth loop group we replace $\mathbb{C}((s))$ with the punctured disc $\{|z| < 1\} - 0$. Again it seems likely that only maps from the punctured disc into X^{aff} that extend are those of the form (12). However one point worth mentioning is that if instead of $L^\times G((s))$ one looks at $(G[z^\pm])[s^\pm]$; that is, polynomial loops in the polynomial loop group then there is a type of Iwahori decomposition; see [4].

For the application to bundles on curves we will need a completeness statement and for that we need a “smaller” version of X^{aff} . To explain “smaller” note one difference between $G((z))$ and say G is that it has two different types of flag varieties. There are thin ones coming from positive parabolic subgroups, e.g. $G((z))/G[[z]]$; these are projective ind-schemes of finite type. Then there are thick flag varieties, e.g. $G((z))/G[z^{-1}]$; these are infinite type schemes and are not proper. From proposition 3.6 we see that each component of the boundary of X^{aff} fibers over a product of a thick and a thin flag variety with fiber generically a finite dimensional reductive group. We would like a version of X^{aff} where the boundary components fiber over the product of two thin flag varieties.

To accomplish this we must look at homogeneous spaces for

$$G^\times((z^{-1})) \times G^\times((z)) := \mathbb{C}^\times \ltimes G((z^{-1})) \times \mathbb{C}^\times \ltimes G((z)).$$

Specifically consider the group $\Delta = \{(\gamma_1(z^{-1}), \gamma_2(z)) \in G^\times((z^{-1})) \times G^\times((z)) \mid \gamma_1(z) = \gamma_2(z)\}$. Then the quotient $G^\times((z^{-1})) \times G^\times((z))/\Delta$ is isomorphic to both $G^\times((z^{-1}))$ and $G^\times((z))$. To construct the embedding we must modify the previous construction of looking at the orbit of the identity because although $G^\times((z^{-1}))$ acts on the representations we have constructed ($\gamma(z) \rightarrow \gamma(z^{-1})$ is essentially a transpose operation) the stabilizer of the identity is not Δ . Instead we look at $G^\times((z^{-1})) \times G^\times((z))$ acting on $W := V(\tilde{\lambda}) \oplus V(\tilde{\lambda})$ where (γ_1, γ_2) acts by $\phi \mapsto \gamma_1 \cdot \phi \cdot \gamma_2^{-1}$ and γ_1, γ_2^{-1} are given by the following operators

$$\begin{aligned} \gamma_1(z) &\mapsto \begin{pmatrix} \gamma_1(z) & 0 \\ 0 & \gamma_1(z^{-1}) \end{pmatrix} \\ \gamma_2^{-1}(z^{-1}) &\mapsto \begin{pmatrix} \gamma_2^{-1}(z) & 0 \\ 0 & \gamma_2^{-1}(z^{-1}) \end{pmatrix} \end{aligned}$$

Then the stabilizer of the identity is $Z(G) \times Z(G) \cdot \Delta$ and we set

$$X_{ind}^{aff} := \overline{G^\times((z^{-1})) \times G^\times((z)) \cdot [id \oplus id]} \subset \mathbb{P}W.$$

X_{ind}^{aff} contains $G^\times((z))/Z(G)$ as a dense open subset and we have

Theorem 3.8.

- (1) X_{ind}^{aff} is smooth and independent of $\tilde{\lambda}$.

- (2) The boundary $X_{ind}^{aff} - G^\times((z))/Z(G)$ consists of $2^{rk(G)+1} G^\times((z^{-1})) \times G^\times((z))$ orbits indexed by the subset $J \subset [1, r+1]$.
- (3) For $J \subset [1, r+1]$ let $P_J^\pm \subset G((z^\pm))$ be the associated parabolics with standard Levi decomposition $P_J = L_J \cdot U_J^\pm$. Set $S(J) = \{(g_1, g_2) \in P_J \times P_J^- : g_1|_{L_J} = g_2|_{L_J}\} = \Delta(L_J) \ltimes (U_J^- \times U_J)$. Also define a group $\ker(J) = \cap_{j \notin J} \ker \alpha_j$ and $T(J) := \ker(J) \times \ker(J)$. Then

$$\begin{aligned} \text{Stab}(e_J) &= T(J) \cdot S(J) \\ X - L^\times G/Z(G) &= \bigsqcup_{J \subset [1, l]} L^\times G \times L^\times G/T(J) \cdot S(J) \end{aligned}$$

In particular, the components of the boundary fiber over the product of two thin flag varieties.

Remark 13. In the smooth setting because $\mathcal{U} \cong \mathcal{U}^-$ there is no asymmetry between in the flag varieties for $L^{sm}G$ and there is no need to make any modification to X^{aff} .

For both X^{aff} and X_{ind}^{aff} we have a cartesian diagram

$$\begin{array}{ccc} L^\times G/Z(G) & \longrightarrow & X^{aff} \\ \downarrow pr_1 & & \downarrow \overline{pr_1} \\ \mathbb{C}^\times & \longrightarrow & \mathbb{A}^1 \end{array} \quad (14)$$

The fiber over 0 we call the special fiber and denoted X_s^{aff} (resp. $X_{ind,s}^{aff}$). For $X_{ind,s}^{aff}$ we have

Theorem 3.9. *An arbitrary morphism*

$$\mathbb{C}((s)) \rightarrow X_{ind,s}^{aff}$$

extends to $\mathbb{C}[[s]]$

Proof. The special fiber $X_{ind,s}^{aff}$ is the union of $r+1$ components and any map from $\mathbb{C}((s))$ lands in one of them. By theorem 3.8 each component maps to a product of thin flag varieties. Over the dense orbit the map is the natural projection from

$$\frac{G^\times((z^{-1})) \times G^\times((z))}{\Delta(L) \cdot (U^- \times U)} \rightarrow G^\times((z^{-1}))/P^- \times G^\times((z))/P$$

where $P^\pm = L \cdot U^\pm$ is a parabolic subgroup of $G^\times((z^\pm))$. Over this orbit the fiber is L and the remaining part of a divisor is a compactification of L ; this follows essentially because we have the Iwahori decomposition for $L((s))$. In other words each component is proper over an ind-projective scheme hence the result. \square

Remark 14. We do not have this statement in the smooth setting because the smooth flag varieties are not complete.

3.3. Line bundles on X^{aff} and Stacky Extension. As in the finite dimensional case we refer to X_0^{aff} as the *open cell* of X^{aff} . So far we have proceeded exactly as in section 2. In the finite dimensional case the above result shows that X_0 is smooth and implies $\text{Pic}(X_0) = 0$. In the ind-scheme setup both of these results are more subtle to prove. In fact \mathcal{U}^- , LG and in particular X_0^{aff} are all not smooth in the sense that they cannot be expressed as a union of smooth schemes; see [11, 5.4]. However there is a more general notion of smoothness for ind-schemes which is called algebraic smoothness by Kumar [20, 4.3]. The loop group is algebraically smooth. This is equivalent to formal smoothness in the sense of being able to lift maps over closed subschemes defined by square zero ideal; see [27, 3.2]. The upshot is X_0^{aff} and consequently X^{aff} are not smooth but are formally smooth. On the other hand $L^{sm}G$ is an infinite dimensional manifold in the sense that each point has a neighborhood isomorphic to a topological vector space, see [26, 3.2].

The initial difficulty in computing $\text{Pic}(X_0^{aff})$ lies in determining $\text{Pic}(\mathcal{U}^-)$. However \mathcal{U}^-/U^- naturally embeds as an ind-variety in LG/L^+G . This allows us to prove

Lemma 3.10. $Pic(\mathcal{U}^- \times \overline{T} \times \mathcal{U}) = 0$

Proof. First we show $Pic(\mathcal{U}^-) = 0$; as varieties $\mathcal{U}^- \cong U^- \times N^-$ where $N^- = \{\gamma \in L^-G \mid \gamma = id \text{ mod } z^{-1}\}$. As $U^- \cong \mathbb{A}^m$ it suffices to show $Pic(N^-) = 0$. The ind-scheme LG/L^+G is a union of Schubert varieties $Y_\mu = \bigcup_{\eta \leq \mu} L^+G\eta L^+G/L^+G$ where \leq is the Bruhat order on affine Weyl group. In particular we get

$$N^- = \bigcup_{\mu \in W^{aff}/W} Y_\mu \cap N^- =: \bigcup_{\mu} N_\mu^-$$

The varieties Y_μ are normal so the singular locus has codim ≥ 2 . Therefor $Pic(\mathcal{U}^-) = Pic(\mathcal{U}_{smooth}^-)$. Working on the smooth locus we can work with Weil divisors and from [19, c.13] we note that $Pic(LG/L^+G) = Pic(Y_\mu) = \mathbb{Z}$ and a generator L_0 for the former restricts to a generator of the latter. There is a section s of L_0 on LG/L^+G such that $Div(s)$ is the complement of N^- . As the restriction of $Div(s)$ to Y_μ generates $Pic(Y_\mu) = \mathbb{Z}$, it follows that $Pic(Y_\mu \cap N^-) = 0$ and consequently that $Pic(N^-) = 0$.

This immediately gives $Pic(\mathcal{U}^- \times \overline{T_{ad,0}^\times}) = 0$. Finally as varieties $\mathcal{U} \cong \prod_{i \in \mathbb{N}} \mathbb{A}^1 = \mathbb{A}^\infty$. The ind-scheme filtration on $\mathcal{U}^- \times \overline{T_{ad,0}^\times}$ induces one on

$$\mathcal{U}^- \times \overline{T_{ad,0}^\times} \times \mathcal{U} = X_0^{aff} = \bigcup_{\mu} N_\mu^- \times U^- \times \overline{T_{ad,0}^\times} \times \mathbb{A}^\infty = \bigcup_{\mu} Z_\mu.$$

So it is enough to show $Pic(Z_\mu) = 0$. For any line bundles on Z_μ fix a finite cover over which the bundle is trivial. Then the transition functions will depend only on a finite number of variables. In particular the bundle is pulled back from some finite dimensional variety of the form $N_\mu^- \times U^- \times \overline{T_{ad,0}^\times} \times \mathbb{A}^l$ which has trivial Picard group. \square

The bundle L_0 is induced from the central extension $\tilde{L}G$ of LG . In fact $\tilde{L}G \rightarrow LG/L^+G$ is a Zariski locally trivial \tilde{L}^+G -bundle. Further, $\tilde{L}^+G \cong \mathbb{C}^\times \times L^+G$ from which we get a character $\chi = pr_2: \mathbb{C}^\times \times L^+G \rightarrow \mathbb{C}^\times$. By definition

$$L_0 = \tilde{L}G \times_{\chi^{-1}} \mathbb{C}^\times$$

See [19] for more details. The line bundle L_0 is important in the connection between bundles on curves because there is morphism $LG/L^+G \rightarrow Bun_G(C)$ on a curve and L_0 is the pull back of the determinant bundle; in fact, the determinant bundle is usually described by descending from L_0 . We can consider L_0 as a line bundle on LG by pulling back via $LG \rightarrow LG/L^+G$.

Proposition 3.11. *In the notation of the proof of 3.7, the irreducible components \overline{D}_i are Cartier divisors that generate $Pic(X^{aff})$. In particular the line bundle L_0 extends to X^{aff} .*

Proof. The D_i are Cartier; in fact there is a maximal parabolic subgroup \mathcal{P}_i and a line bundle L_i on G^{aff}/\mathcal{P}_i such that D_i is the inverse image of the vanishing of a section σ_i on G^{aff}/\mathcal{P}_i . Moreover choosing a highest weight representation V_i such that \mathcal{P}_i stabilizes the class of a highest weight vector v_i then there is a morphism

$$G^{aff} \rightarrow G^{aff}/\mathcal{P}_i \rightarrow \mathbb{P}V_i \quad g \mapsto g\mathcal{P}_i \mapsto [g.v_i]$$

and the divisor D_i is the divisor associated to the pull back of $\mathcal{O}(1)$ on $\mathbb{P}V_i$. Define

$$X^{aff}(i) := \overline{G^{aff} \times G^{aff} \cdot [id]} \in \mathbb{P}[V_i \hat{\otimes} V_i^*]$$

Then there is a line bundle, which by abuse of notation, we also call L_i given by restriction of $\mathcal{O}(1)$ to $X^{aff}(i)$. Finally, we claim there is a morphism $X^{aff} \rightarrow X^{aff}(i)$ and \overline{D}_i is the divisor associated to the pull back of L_i to X^{aff} . To see this note that $X^{aff}(i)$ is also covered by open sets of the form $\mathcal{U}^- \overline{T_{ad,0}^\times}(i) \mathcal{U}$ where

$$\overline{T_{ad,0}^\times}(i) = \overline{T_{ad,0}^\times} \cap \{v_i \otimes v_i^* \neq 0\}$$

so a unique $G^{aff} \times G^{aff}$ equivariant morphism is determined by giving a map

$$\overline{T_{ad,0}^\times} \rightarrow \overline{T_{ad,0}^\times}(i)$$

and this map exists because the fan of $\overline{T_{ad,0}^\times}$ is always a refinement of the fan of $\overline{T_{ad,0}^\times}(i)$. The final point follows because the fan of $\overline{T_{ad,0}^\times}$ is given by the inequalities

$$\begin{aligned} -\alpha_0(t) &\geq 0 \\ &\vdots \\ -\alpha_r(t) &\geq 0 \end{aligned}$$

whereas the fan for $\overline{T_{ad,0}^\times}(i)$ has equalities given by replacing some of the above with inequalities of the form $-\sum_i n_i \alpha_i(t) \geq 0$. \square

As in the finite dimensional case we can construct a stacky extension. Consider the ind group $H = G^{aff} \times (\mathbb{C}^\times)^{r+1}$. As before we have a cone with support in the negative Weyl chamber. The dual cone c^\vee is generated by the fundamental weights $\pm \tilde{\omega}_i$ and e_i where $e_i: (\mathbb{C}^\times)^{r+1} \rightarrow \mathbb{C}^\times$ is the i th projection. By the 3.7 we get an H -embedding

$$\overline{H}_c \subset \mathbb{P}[\tilde{\lambda}, \tilde{\lambda} \pm \tilde{\omega}_i, \tilde{\lambda} + e_i]$$

Let $A \subset V_T \otimes \mathbb{R}$ be the negative Weyl alcove. It corresponds to a strongly convex polyhedral cone in $V_{T^\times} \otimes \mathbb{R}$; let u_1, \dots, u_{r+1} be generators for the rays of A . Consider the homomorphisms of lattices

$$\beta: \mathbb{Z}^{r+1} \xrightarrow{e_i \mapsto u_i} V_{T^\times}.$$

Then associated to the stacky fan (c, β) we get a surjection $(\mathbb{C}^\times)^{r+1} \xrightarrow{\pi} T^\times$ and consequently a homomorphism

$$T_\beta := (\mathbb{C}^\times)^{r+1} \xrightarrow{\pi, id} T^\times \times (\mathbb{C}^\times)^{r+1} \subset H.$$

The compactification \overline{H}_c carries an action of $H \times H$ so we get an action of $T_\beta \times T_\beta$. Identify T_β with $T_\beta \times T_\beta / \Delta(T_\beta)$. As before

Definition 3.12. The stacky version of wonderful compactification is

$$\mathcal{X}^{aff} = \overline{L^\times G} := [\overline{H}_c / T_\beta]$$

It is a completion in the sense that it is finite over X^{aff} . The proof of 2.9 remains valid and gives

Theorem 3.13.

- (a) \mathcal{X}^{aff} is independent of λ .
- (b) $\mathcal{X}^{aff} - \mathcal{X}_0^{aff}$ is of pure codimension 1 and we have an exact sequence

$$0 \rightarrow \mathbb{Z}^r \rightarrow \text{Pic}(\mathcal{X}^{aff}) \rightarrow Z(G) \rightarrow 0$$

where the subgroup \mathbb{Z}^r is generated by the irreducible components of $\mathcal{X}^{aff} - \mathcal{X}_0^{aff}$.

- (c) The boundary $\mathcal{X}^{aff} - L^\times G$ consists of $r+1$ normal crossing divisors D_1, \dots, D_{r+1} and the closure of the $L^\times G \times L^\times G$ -orbits are in bijective correspondence with subsets $I \subset [1, r+1]$ in such a way that to I we associate $\cap_{i \in I} D_i$.
- (d) Let u_1, \dots, u_{r+1} be generators of the rays of the Weyl alcove and M be the monoid they generate. Any $G^{aff} \times G^{aff}$ equivariant $\mathcal{X}' \rightarrow \mathcal{X}^{aff}$ determines and is determined by a fan supported in the negative Weyl chamber whose lattice points lie in M .

Remark 15. Note for the proof of (b) that although $\overline{T_0} \not\cong \overline{T_0^\times}$ it nevertheless holds that they have the same Picard group $\cong Z(G)$. This follows from explicit computation with the fan of $\overline{T_0^\times}$.

Remark 16. One can see that 2.9 follows formally from 2.1. In the same way the previous result follows formally from 3.7. The latter holds for both the algebraic and smooth the loop group and we conclude the same for 3.13.

Remark 17. One can also work with the polynomial loop group $L_{poly}G = G[z^\pm]$; it is an ind-group scheme of finite type. Everything goes through as before and we get an embedding of $L_{poly}G$ or rather of $\mathbb{C}^\times \ltimes L_{poly}G$. Theorems 3.7, 3.13 remain valid.

Remark 18. So far we have not discussed the ind-structure on X^{aff} . However in section 4.4 we do define a projective ind-scheme $\mathbb{P}^{ind} \subset \hat{\mathbb{P}}$ and show that $X^{aff} \subset \mathbb{P}^{ind}$ and further that $L^\times G \rightarrow \mathbb{P}^{ind}$ is a morphism of ind-schemes.

4. REPRESENTATION THEORY OF LT

An interesting aspect of the representation of loop groups is the appearance of a new intermediate abelian group. More precisely, instead of asking for a maximal torus $T \subset LG$, we can ask for a maximal abelian subgroup $H \subset LG$. The standard choice is $H = LT$ but there are others choices, see for example [24, 3.6]. What is interesting is the restriction of the central extension $\tilde{L}G$ to $\tilde{L}T$ is non trivial and has interesting irreducible representations. Further, much like we can gain insight into irreducible representations of G by decomposing according the action of T , we can understand irreducible representations of $\tilde{L}G$ by decomposing them according to the action of $\tilde{L}T$. Indeed positive energy representations for $\tilde{L}G$ break up into a finite sum of positive energy representations of $\tilde{L}T$.

In this section we study positive energy representations of central extensions of $L^\times G$ and $L^\times T$. We first focus on $L^\times T$ or rather its central extension which is generated by two subgroups

$$\mathbb{C}^\times \ltimes T \tilde{\times} V_T \subset \widetilde{L^\times T} \supset (\widetilde{LT/T})_0$$

where $T \tilde{\times} V_T$ is the pull back of the central extension \mathbb{C}_c^\times :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{C}_c^\times & \longrightarrow & T \tilde{\times} V_T & \longrightarrow & T \times V_T \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \mathbb{C}_c^\times & \longrightarrow & \tilde{L}T & \longrightarrow & LT \longrightarrow 0 \end{array}$$

and similarly for $(\widetilde{LT/T})_0$. In addition, we would like to mention an interesting connection with combinatorics. We shall see shortly that the group structure on $\tilde{L}T$ depends on a choice of a positive definite symmetric form Q on the Lie algebra \mathfrak{t} of T . In the case T sits inside a semisimple group G , then this form comes from the Killing form. The closure of the torus in the wonderful compactification of LT is given by the cone on what is known in combinatorics as the Voronoi decomposition associated to Q . For the representation of LG we will use a more algebraic approach using quotients of Verma modules.

In general, writing down a complete description of a positive energy representation requires some work. See [24, ch.9,10,11], for some examples. We give one construction for LT which is fairly explicit and follows [26]. Let us begin by pinning down the group structure of $T \tilde{\times} V_T$. We can determine the structure on the central extension by looking at the case when T sits inside a simple group G . In this setting we have a linear map $V_T \rightarrow \Lambda_T$, $\eta \mapsto \langle \eta, \cdot \rangle$ given by the Killing form. The affine Weyl group $W^{Aff} = W \ltimes V_T$ has a natural action on the characters of $T^{aff} := T^\times \times \mathbb{C}_c^\times = \mathbb{C}^\times \times T \times \mathbb{C}_c^\times$. This action is computed in the proof of [20, 13.1.7]. We identify characters with their derivatives so that if we have an element $t^{aff} = (e^r, e^t, e^c) \in T^{aff}$ then a character $\tilde{\lambda} = (n, \lambda, h)$ evaluates to

$$\lambda(t^{aff}) = e^{nr + \lambda(t) + hc}.$$

With this notation the action of $\eta \in V_T \subset W^{aff}$ is

$$\begin{aligned} \tilde{\lambda} &\xrightarrow{\eta} \eta \cdot \tilde{\lambda} = \tilde{\lambda} + \left(-\lambda(\eta) + \frac{h}{2} \langle \eta, \eta \rangle, \quad h \langle \eta, \cdot \rangle, \quad 0 \right) = \left(n - \lambda(\eta) + \frac{h}{2} \langle \eta, \eta \rangle, \quad \lambda + h \langle \eta, \cdot \rangle, \quad h \right) \\ \begin{bmatrix} e^r \\ e^t \\ 1 \end{bmatrix} &\mapsto \exp \left(\left[n - \lambda(\eta) + \frac{h}{2} \langle \eta, \eta \rangle \right] \cdot r + [\lambda(t) + h \langle \eta, t \rangle] \right) \\ &\mapsto \exp \left([n - \lambda(\eta)] \cdot r + \lambda(t) + h \cdot [\langle \eta, t \rangle + \frac{1}{2} \langle \eta, \eta \rangle r] \right) \end{aligned}$$

We can use this to understand conjugation in the central extension, for the action of η can also be interpreted as

$$\begin{aligned} Taff &\xrightarrow{\eta \circ (-) \circ \eta^{-1}} Taff \xrightarrow{\tilde{\lambda}} \mathbb{C}^\times \\ \begin{bmatrix} e^r \\ e^t \\ 1 \end{bmatrix} &\mapsto \begin{bmatrix} e^r \\ e^{t - \langle \eta, \rangle r} \\ c(\eta, t, r) \end{bmatrix} \mapsto \exp \left(nr + \lambda(t) - \lambda(\eta)r + hc(\eta, t, r) \right) \\ &\mapsto \exp \left([n - \lambda(\eta)] \cdot r + \lambda(t) + h \cdot c(\eta, t, r) \right) \end{aligned}$$

We are using the semi-direct product to compute the middle row above. Comparing we see

$$c(\eta, t, r) = \langle \eta, t \rangle + \frac{\langle \eta, \eta \rangle}{2} r$$

One can check that $c(\eta, t, r)$ is a co-cycle in the sense that it obeys the associative law of group multiplication. In fact $c(\eta, t, r)$ determines the group law on $\mathbb{C}^\times \ltimes T \tilde{\times} V_T$ because topologically $\mathbb{C}^\times \ltimes T \tilde{\times} V_T = \mathbb{C}^\times \times T \times V_T \times \mathbb{C}_c^\times$ and denoting elements by $(u, H, \eta, w)^T$, the group structure has the form:

$$\begin{aligned} \begin{bmatrix} u_1 \\ H_1 \\ \eta_1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ H_2 \\ \eta_2 \\ 1 \end{bmatrix} &= \begin{bmatrix} u_1 \\ H_1 \\ 1 \\ 1 \end{bmatrix} \cdot \eta_1 \cdot \begin{bmatrix} u_2 \\ H_2 \\ \eta_2 \\ 1 \end{bmatrix} \cdot \eta_1^{-1} \eta_1 \\ &= \begin{bmatrix} u_1 \\ H_1 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} u_2 \\ \eta_1^{-1}(u_2)H_2 \\ \eta_1\eta_2 \\ c(\eta_1, H_2, \rho_2) \end{bmatrix} = \begin{bmatrix} u_1 u_2 \\ \eta_1^{-1}(\rho_2)H_1 H_2 \\ \eta_1 \eta_2 \\ c(\eta_1, H_2, \rho_2) \end{bmatrix} \end{aligned}$$

Thus we can get a group structure for every positive definite symmetric form Q via

$$c_Q(\eta, H, u) = Q(\eta, H) + u \frac{Q(\eta, \eta)}{2}$$

In fact this makes sense even if Q is not positive definite. However when we look at representations of these groups we want the weight spaces of u to be bounded below and we also want the weight spaces to be finite dimensional; these conditions fail if Q is not positive definite.

4.1. Representation of $\mathbb{C}^\times \ltimes T \tilde{\times} V_T$. In [26], Segal describes a representation of $\mathbb{C}^\times \ltimes \tilde{L}T$ on a vector space of the form $V_1 \otimes V_2$ where V_1 is a countable dimensional representation of $\mathbb{C}^\times \ltimes T \tilde{\times} V_T$ and V_2 is an uncountable dimensional representation of $(LT)_0/T$. This produces a positive energy representation of $\tilde{L}T$. We describe both representations but focus first on V_1 because this is all that is necessary to understand the closure

$$\overline{T^\times} \subset \overline{L^\times T}$$

More precisely we'll see that the orbit of the identity in $\mathbb{P}End(V_2)$ under $(LT)_0/T$ is closed and hence doesn't contribute to the boundary.

Fix a positive definite symmetric form Q on V_T ; we view this as a linear map $V_T \rightarrow \Lambda_T$. From the previous section this determines a group $\tilde{Q} := \mathbb{C}^\times \ltimes T \tilde{\times} V_T$. We describe a representation of \tilde{Q} on $L = \oplus_{\chi \in Q(V_T)} L_\chi$. In fact following the wonderful compactification recipe we look at the $\tilde{Q} \times \tilde{Q}$ orbit of the identity in $End(L)$. Let Z_Q be the finite group which is the kernel of $Q: V_T \rightarrow \Lambda_T$. The embedding has

$$\tilde{Q} \times \tilde{Q} / (Z_Q \mathbb{C}_c^\times \times Z_Q \mathbb{C}_c^\times) \Delta(\tilde{Q}) = \mathbb{C}^\times \ltimes T/Z_Q \times V_T$$

as a dense open subset set. The connected components are indexed by V_T and the connected component of the identity is the closure of $T^\times = \mathbb{C}^\times \times T$. The representation is simple to describe; let v_μ be a weight

vector with weight μ , then:

$$\begin{aligned} \eta.v_\mu &= v_{\mu+Q(\eta)} \\ t.v_\mu &= \mu(t)v_\mu \\ u.v_\mu &= u^{Q(\mu,\mu)/2}.v_\mu \end{aligned} \tag{15}$$

Remark 19. One can check that this indeed defines a representation; a verification of this appears in [26, pg.306]. Further any highest weight representation of LG decomposes into a finite direct sum of LT representations. One important consequence is that for any co-character $\eta \in V_{T^\times}$ such that the composition $\mathbb{C}^\times \xrightarrow{\eta} T^\times = \mathbb{C}^\times \times T \xrightarrow{pr_1} \mathbb{C}^\times$ is given by a positive integer has the property that

$$\lim_{s \rightarrow 0} \eta(s) \text{ exists}$$

as an element of $\mathbb{P}(V \otimes V^*)$. To see this it is enough to check the function $\mu \rightarrow \langle \mu, \eta \rangle$ from the weights of V to \mathbb{Z} has a well defined minimum value. This follows because the last line of (15) shows the function $\mu \rightarrow \langle \mu, \eta \rangle$ is quadratic with leading order coefficient positive. Further the minimum is achieved at a finite number of weight spaces μ .

We can identify the image of T^\times in $End(L)$ as diagonal matrices with infinitely many nonzero terms. Such matrices we can identify with $\hat{L} = \prod_\mu L_\mu$. With this translation we have a representation of $T^\times \rightarrow \hat{L}$ and the toric variety we are interested in is $\overline{T^\times} := T^\times \cdot \left[\prod_\mu 1 \right] \in \mathbb{P}\hat{L}$.

Let us now briefly describe the representation of $(LT)_0/T$ on V_2 . We do this for $T = \mathbb{C}^\times$ but the construction clearly extends to higher rank. We have

$$\begin{aligned} LT(R) &= L^{<0}T(R) \times T(R) \times V_T \times L^{>0}T(R) \\ L^{<0}T(R) &= \{id + \sum_{i=-n}^{-1} r_i z^i | r_i \in R_{nilpotent}\} \\ L^{>0}T(R) &= \{id + \sum_{i=1}^{\infty} r_i z^i | r_i \in R\} \end{aligned}$$

We see $(LT)_0/T = L^{<0}T(R) \times L^{>0}T(R)$ and is not reduced. Let W be the \mathbb{C} -vector space $\oplus_{i>0} \mathbb{C}z^i \subset L^{>0}T(\mathbb{C})$. The representation of $(LT/T)_0$ is on the vector space $V_2 = \widehat{Sym}(W) = \mathbb{C}[x_1, x_2, \dots]$ where hat means we allow infinite sums of elements in $\mathbb{C}[x_1, x_2, \dots]$.

The action of $\gamma(z) \in L^{>0}T$ is given by multiplication by $\exp(\log \gamma(z))$ where $\log \gamma(z) = \sum_{i>1} a_i z^i$ and z^i acts by multiplication by x_i . For example if $\gamma = \exp(z)$, then $\log \gamma = z$ and if $f \in Sym(W)$ then $\log \gamma$ is given by $f \mapsto x_1 f$. Thus γ acts by multiplication by $\exp(x_1)$. If $\gamma \in L^{<0}T$ then because $\gamma = 1 + n$ where n is nilpotent we can write $\log \gamma = \sum_{i=1}^m a_i z^{-i}$. Then $\log \gamma$ acts by $\sum_{i=1}^m a_i \frac{\partial}{\partial x_i}$ and exponential of this operator gives the action of γ . An important point to note is that this extends to a *lowest* weight representation of LG .

As before let $\overline{(LT/T)_0}$ be the closure in $\mathbb{P}End[\widehat{Sym}(W)]$ of the orbit of the identity. We claim over reduced R that $\overline{(LT/T)_0} = (LT/T)_0$. Indeed over reduced R we just need to consider $L^{>0}T$ and a point in the boundary can be understood as a diagram

$$\begin{array}{ccc} \text{Spec } R((s)) & \xrightarrow{\phi} & L^{>0}T(R) \\ \downarrow & & \downarrow \\ \text{Spec } R[[s]] & \longrightarrow & \mathbb{P}End_R(\widehat{Sym}(W)) \end{array}$$

Think of ϕ as an element $1 + \sum_{i \geq 1} \phi_i(s) z^i$ where $\phi_i(s) \in R((s))$. If ϕ_1 contains negative powers of s then ϕ will act by multiplication by an element where the coefficients of x_1^n will have arbitrary high negative exponents of s . Consequently the limit $s \rightarrow 0$ does not exist in $\mathbb{P}End(V_2)$. It follows $\phi_1(s) \in R[[s]]$. Continuing inductively we see all $\phi_i \in R[[s]]$.

In the the smooth setting, where $T = (S^1)^n$ and $LT = \text{Map}_{C^\infty}(S^1, T)$ then again $(LT/T)_0$ breaks up into a negative and positive piece that act on $\widehat{\text{Sym}}(W)$ just as described above. In this case the action of $L^{<0}T$ can be described as z^{-i} maps $x_i \rightarrow x_i + 1$ and all other $x_j \rightarrow 0$. In this case we can apply a similar argument as above. Namely, $\phi_i(s)z^{-i} \cdot (x_i)^n = (x_i + \phi_i(s))^n$ so if ϕ_i has negative exponents the limit $s \rightarrow 0$ will not exist. Thus in the smooth setting the boundary of LT is entirely determined by the closure of the torus.

We now turn to describing the $\text{Spec } \mathbb{C}$ points of the boundary of $\overline{T^\times}$. As we have seen it is enough to consider the closure of the torus. We would like to express the fan for $\overline{T^\times}$ in terms of combinatorics associated to Q . We recall the relevant combinatorics now.

4.2. Voronoi and Delaunay Subdivisions and the fan. Let W be a real vector space equipped with an inner product Q . Given a finite set of points $S \subset W$ one can form the Voronoi diagram or Voronoi partition associated to S . This is partition $W = \cup_{s \in S} C_s$ where

$$C_s = \{w \in W | Q(s, w) \leq Q(s', w) \forall s' \in S - s\}$$

We can replace S with a regular lattice and apply the same procedure to get tiling of W . For example if $W = \mathbb{R}^2$ and $Q = I$ and we consider the lattice \mathbb{Z}^2 then we get a tiling by squares centered around the points of \mathbb{Z}^2 . These tilings are well studied and have many nice properties; see [30].

Dual to Voronoi diagrams is the notion of Delaunay subdivision. Let $L \subset W$ be a lattice. For any $p \in W$ let $r(p)$ be the minimum distance such that $B(p, r(p)) \cap L \neq \emptyset$ where $B(p, r) = \{w \in W | Q(p, w) \leq r\}$. Let $P(p)$ be the convex hull of all lattice points inside $B(p, r(p))$. Then we get a subdivision of W via $W = \cup_{p \in W} P(p)$. Alternatively, the convex hull of $p_1, \dots, p_n \in L$ is in the subdivision if and only if $C_{p_1} \cap \dots \cap C_{p_n} \neq \emptyset$.

It is known that Delaunay subdivisions can be computed using a certain lifting property. Namely consider the graph $(id, Q): L \rightarrow L \times \mathbb{R}$ and let $C(Q, L) \subset W \times \mathbb{R}$ be the convex hull of the lifted points. Let $DC(Q, L)$ be the subset of $C(Q, L)$ consisting of all facets that can be seen from “ $-\infty$ ”; Let $\pi: W \times \mathbb{R} \rightarrow W$ be the projection. The Delaunay subdivision consists of all facets of the form $\pi(F)$ where $F \in DC(Q, L)$.

One can get the Voronoi subdivision using similar considerations. Note that the image of $id, Q: W \rightarrow W \times \mathbb{R}$ is a paraboloid \mathcal{Q} . Let \mathcal{Q}_{p_1} be the tangent plane to \mathcal{Q} at the lifted lattice point $(p_1, Q(p_1, p_1))$. It turns out that the projection of $\mathcal{Q}_{p_1} \cap \mathcal{Q}_{p_2}$ in W is the hyperplane of points equidistant between p_1, p_2 . It is not difficult to see that C_{p_1} is a polytope whose supporting hyperplanes consist of a subset of the hyperplanes of points equidistant from p_1 and other lattice points.

Let C_s^0 be defined by replacing \leq with $<$ in the definition of C_s . Then $C_0^0 = \{p \in W | P(p) = 0\}$.

Now let us explain the connection between this combinatorics and $\overline{T^\times} \subset \overline{L^\times T}$.

Let \mathcal{F} be the fan of $\overline{T^\times}$. We have $\mathcal{F} \subset V_{T^\times} \otimes \mathbb{R} = \mathbb{R} \oplus V_{T, \mathbb{R}}$. We can now state

Theorem 4.1. *The fan \mathcal{F} is contained in $V_{T, \mathbb{R}} \oplus \mathbb{R}_{>0}$. For a > 0 identify $V_{T, \mathbb{R}}$ with $a \oplus V_{T, \mathbb{R}}$; there is a lattice $L_a \subset V_{T, \mathbb{R}}$ such that \mathcal{F} is the cone on the Voronoi diagram associated to $(L_a, V_{T, \mathbb{R}}, Q)$.*

Proof. Let $(b, \eta) \in V_{T^\times} = \mathbb{Z} \oplus V_T$ be a co-character. Fix a coordinate s on \mathbb{C}^\times then composing

$$\mathbb{C}^\times \xrightarrow{(b, \eta)} T^\times \subset \mathbb{P}\hat{L}$$

we can express the image as

$$\prod_{Q(\alpha) \in Q(V_T)} s^{F_{b, \eta}(Q(\alpha))} \in \mathbb{P}\hat{L}$$

$$F_{b, \eta}(Q(\alpha)) = s^{\frac{b}{2}\alpha Q\alpha + \eta Q\alpha}$$

This follows from (15). The limit $s \rightarrow 0$ only exists if $b > 0$. Moreover, if $b > 0$, then the limit exists for any value of η . This gives the first claim.

Fix now $a \in \mathbb{Z}_{>0}$ and consider the 1 parameter subgroup $\text{diag}(s^{F_{a, \eta}(Q(\alpha))})$. Set

$$p_\eta = \lim_{s \rightarrow 0} \prod (s^{F_{a, \eta}(Q(\alpha))}) \in \mathbb{P}\hat{L} = \mathbb{P} \prod_{\mu \in Q(V_T)} L_\mu.$$

Consider the function $F_{a,\eta}$ as a function on V_T ; it is quadratic and has a unique global minimum on $V_{T,\mathbb{R}}$ therefore there are only finitely many co-characters $\eta_1, \dots, \eta_n \in V_T$ where $F_{a,\eta}$ attains its minimum, therefore

$$p_\eta = \prod_{Q(\alpha) \in Q(V_T)} (f(\alpha))$$

$$f(\alpha) = \begin{cases} 1 & \text{if } \alpha \in \{\eta_1, \dots, \eta_n\} \\ 0 & \text{otherwise} \end{cases}$$

So in fact $p_\eta \in \mathbb{P}L$. We can identify the cones of \mathcal{F} with the polytopes $C_\eta = \{\eta' \in V_T | p_\eta = p_{\eta'}\}$.

Consider the subset $L_a = \{\min F_{a,\eta} \in V_{T,\mathbb{R}} = a \oplus V_{T,\mathbb{R}} | \eta \in V_T\}$. A basic application of the quadratic formula shows that L_a is a lattice and we have identified the cones of the fan with Voronoi subdivision associated to L_a . \square

Remark 20. The fan \mathcal{F} contains infinitely many cones one can compute much about the fan in terms of Q . For example one can compute supporting hyperplanes for the polytopes in the Voronoi subdivision, dimension of a cone in terms of number of minimums of $F_{a,\eta}$ on V_T , etc. For precise statements see [30]

If we had worked with a the full representation $V_1 \otimes V_2$ this would had the effect of adding a positive constant to the function $F_{a,\eta}$ which doesn't effect where minimums occur.

Further, the above representations extend to representations of G^{aff} but they correspond to a dominant but in general not a regular dominant weight. If you look at the closure of T^\times inside a regular dominant representation of G^{aff} you do not get a fan that is the cone on a Voronoi subdivision associated to the Killing form. This is because a regular irreducible representation of $\tilde{L}G$ does not stay irreducible under the action of $\tilde{L}T$. In fact for a fixed form Q the irreducible representations of $\tilde{L}T$ are indexed by the finite group $\Lambda_T/Q(V_T)$. One can see just from looking at the character of a representation that in a regular representation of G^{aff} all the irreducible representations of $\tilde{L}^\times T$ appear as a direct summand.

We computed the toric variety of the closure of the torus in the representation corresponding to $0 \in \Lambda_T/Q(V_T)$. The closure in the representations corresponding to nonzero elements in $\Lambda_T/Q(V_T)$ can also be expressed a Voronoi diagram associated to Q but the lattice L_a will be different. In fact one can express the fan of the closure of T^\times in a regular G^{aff} -representation as a kind of intersection of the fans coming from all the $\tilde{L}^\times T$ representations. However with a regular dominant representation of G^{aff} there are more straightforward descriptions of the fan of $\overline{T^\times}$; e.g. as the cone on the Weyl alcove decomposition.

4.3. Example. Consider the case $T = \mathbb{C}^\times$ then the form Q above is a positive integer; we take $Q = 2$. This choice comes from the killing form on $\mathbb{C}^\times \subset SL_2$. Let us consider the Voronoi subdivision associated to \mathbb{Z} in the normed vector space (\mathbb{R}, Q) . Clearly the set of points closes to $n \in \mathbb{Z}$ consists of the line segment $[n - 1/2, n + 1/2]$; in the notation of 4.2, $C_n = [n - 1/2, n + 1/2]$. The fan of $T^\times \subset \overline{L^\times \mathbb{C}^\times} / \pm 1$ is shown in figure 1 where we have identified \mathbb{R} with $\mathbb{R} \oplus 1$.

To differentiate $T = \mathbb{C}^\times$ from the \mathbb{C}^\times in the semidirect product we write $T = GL_1$. The associated toric variety $\overline{T^\times}$ has a map to \mathbb{A}^1 extending the projection $\mathbb{C}^\times \times GL_1 \xrightarrow{pr_1} \mathbb{C}^\times$ and fits into the following diagram

$$\begin{array}{ccccc} \mathbb{C}^\times \times GL_1 & \hookrightarrow & \overline{T^\times} & \longleftarrow & \bigcup_{j \in \mathbb{Z}} \mathbb{P}_j^1 \\ \downarrow & & \downarrow & & \downarrow \\ \mathbb{C}^\times & \hookrightarrow & \mathbb{A}^1 & \longleftarrow & 0 \end{array}$$

4.4. Representations of G^{aff} . Here we discuss some properties of highest weight representations of G^{aff} . We use the Verma module approach to construct representations of G^{aff} but we do give a rough comparison with the previous construction.

Starting from a regular dominant weight $\tilde{\lambda} = (0, \lambda, l)$, which must satisfy $\lambda(\theta^\vee) \leq l$, we consider first the representation of $V(\lambda)$ of \mathfrak{g} . Extend this to a representation of $\mathfrak{g}[[z]] \oplus \mathbb{C}_c$ by making \mathbb{C}_c act by multiplication by l and making $z\mathfrak{g}[[z]]$ act trivially. Induce this to a representation of $\mathfrak{g}((z)) \oplus \mathbb{C}_c$ on $V(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(z^{-1}\mathfrak{g}[z^{-1}])$; where $\mathcal{U}(z^{-1}\mathfrak{g}[z^{-1}])$ is the universal enveloping algebra. This module has a unique

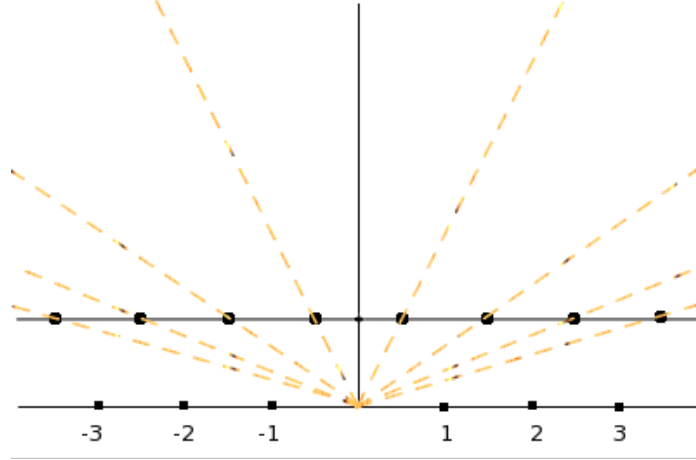


FIGURE 1. Cone on the Voronoi subdivision. The lattice \mathbb{Z} is shown in square dots and the lattice points of the Voronoi subdivision is shown in circle dots. The rays of the fan are shown in dashed dots. Note only a finite number of rays are drawn but every point in the upper half plane lies on a ray or in between two rays.

maximal proper submodule \mathcal{Z}_λ ; it is generated by $v_\lambda \otimes (X_\theta \otimes z^{-1})^{l-\lambda(H_\theta)+1}$ where θ is the highest root of \mathfrak{g} ; see [1]. The quotient is an irreducible representation with highest weight $\tilde{\lambda}$

$$V(\tilde{\lambda}) = V(\lambda) \otimes_{\mathbb{C}} \mathcal{U}(z^{-1}\mathfrak{g}[z^{-1}]) / \mathcal{Z}_\lambda$$

We can see the connection between these description by looking at the weight vectors. Starting from the Verma module perspective a weight vector looks like $Y_{\alpha_{i_1}}^{j_1}(-n_1) \cdots Y_{\alpha_{i_m}}^{j_m}(-n_m)$. The corresponding weight for this vector is (n, μ, l) where

$$n = \sum_{k=1}^m j_k \cdot (-n_j) \quad \mu = \sum_{k=1}^m j_k \alpha_{i_j}$$

therefor, comparing with the previous approach, this must lie in the space $\mathbb{C} \cdot [\mu] \otimes \widehat{\mathbb{C}[x_1, x_2, \dots]}$ and must be of the form $[\mu] \otimes f(\dots, x_i, \dots)$ where f is a homogeneous polynomial of degree $d = n - Q(\mu, \mu)/2$; the grading on $\mathbb{C}[x_1, x_2, \dots]$ has $\deg x_i = i$.

Let us also mention that following convention the representations we constructed for $\tilde{L}^\times T$ were lowest weight representations. Thus to make a comparison between these two approaches one must dualize one of the representations. Let us consider the example of the basic representation of LSL_2 with highest weight $(0, 0, 1)$. As a vector space this representation is the dual of

$$\bigoplus_{n \in \mathbb{Z}} \mathbb{C} \cdot [2n] \otimes V_2$$

where $\mathbb{C} \cdot [2n]$ is the representation of GL_1 given by the character $[2n]$ and $V_2 = \widehat{\mathbb{C}[x_1, x_2, \dots]}$. The weight spaces with multiplicity are shown in figure 2 as well as the action of the lowering operators $Y =$

$$a^{(1)} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \text{ and } -\alpha^{(0)} = X_\theta \otimes z^{-1} = \begin{pmatrix} 0 & z^{-1} \\ 0 & 0 \end{pmatrix}.$$

We now define a certain ind scheme in $\hat{\mathbb{P}}$. We discuss first some preliminaries on how $\gamma \in LG$ acts on $V(\tilde{\lambda})$. Using the $\mathbb{C}^\times \subset L^\times G$ in the semidirect product we can decompose the representation into weight spaces for \mathbb{C}^\times .

$$V(\tilde{\lambda}) = \bigoplus_{j \geq 0} V_j$$

in the theory of positive energy representations these are called the energy spaces. In figure 2 these are the rows. The V_j are in general reducible representations of G . In an appropriate basis G acts on $V(\tilde{\lambda})$

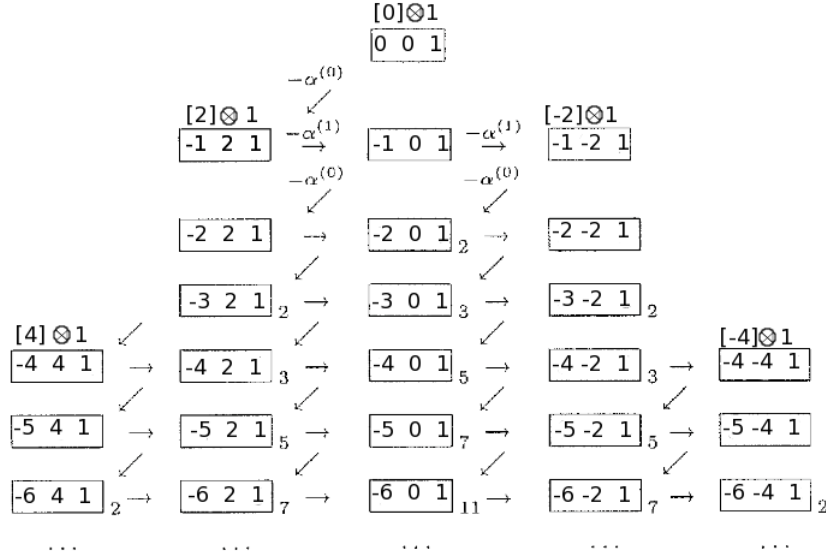


FIGURE 2. A picture of the weight diagram for the basic representation of LSL_2 . The numbers enclosed in boxes are weights for the torus $\mathbb{C}^\times \times GL_1 \times \mathbb{C}^\times$. The subscripts on the boxes indicate the multiplicity. The labels $[n] \otimes 1$ indicate what vector the weight corresponds in the description as $V_1 \otimes V_2$. For example the box that reads $(-1, 0, 1)$ corresponds to vector $[0] \otimes x_1$. This figure was adapted from [12, 2.5.13]

by operators

$$\begin{pmatrix} A_0 & & & 0 \\ & A_1 & & \\ & & A_2 & \\ 0 & & & \ddots \end{pmatrix} \quad A_j \in SL(V_j)$$

note that the dimension of the spaces V_j are monotonically increasing so the blocks A_i are getting larger and larger. Construct an ind-scheme by setting \mathbb{P}_0^{ind} to be the subspace of upper triangular matrices with respect to the above block decomposition:

$$\mathbb{P}_0^{ind} = \left\{ \begin{pmatrix} A_{0,0} & A_{0,1} & A_{0,2} & \dots \\ & A_{1,1} & A_{1,2} & \dots \\ & & A_{2,2} & \dots \\ 0 & & & \ddots \end{pmatrix} \middle| A_{i,j} \in \text{hom}(V_j, V_i) \text{ arbitrary, but not all zero} \right\}$$

For each $n \geq 1$ define \mathbb{P}_n^{ind} to be set of block of matrices where at the k th block column you allow blocks $B_{k+1,k}, \dots, B_{k+n(k+1),k}$ to be nonzero as well; the difference in name of the block is just to emphasize how they are being added. For example

$$\mathbb{P}_1^{ind} = \left\{ \begin{pmatrix} A_{0,0} & \dots & & & \\ B_{0,1} & A_{1,1} & \dots & & \\ 0 & B_{2,1} & A_{2,2} & \dots & \\ \vdots & B_{3,1} & B_{3,2} & A_{3,3} & \dots \\ & 0 & B_{4,2} & B_{4,3} & A_{4,4} & \dots \\ & \vdots & B_{5,2} & B_{5,3} & B_{5,4} & A_{5,5} & \vdots \\ & & 0 & B_{6,3} & B_{6,4} & B_{6,5} & A_{6,6} \end{pmatrix} \right\}$$

One can compose elements in \mathbb{P}_n^{ind} and for any m, n there is a $k = k(m, n)$ such that $\mathbb{P}_n^{ind} \cdot \mathbb{P}_m^{ind} \subset \mathbb{P}_k^{ind}$. To bring in loop groups note that any $\gamma \in LG$ is a finite product

$$\gamma = \prod_{j=k}^m U_\alpha(b) \quad b \in \mathbb{C}((z))$$

so to show LG embeds in \mathbb{P}_0^{ind} it suffices to show $U_\alpha(b)$ lands in some \mathbb{P}_n^{ind} . Now

$$U_\alpha(b) = \prod_{j=-k}^{\infty} \exp(b_j \cdot X_\alpha \otimes z^j) \quad b_j \in \mathbb{C}.$$

so it suffices to show that for every energy space V_l that there is an integer $i = i(l)$ that depends at most linearly on l such that

$$U_\alpha(b) \cdot V_l \subset \oplus_{j \leq l+i} V_j$$

This follows from parabolic nature of the weight spaces of $V(\tilde{\lambda})$; see figure 2. Indeed for a fixed j there is a constant c such that for $n \geq c$ the action of X_α^n is trivial on V_j . From the parabolic nature of the weights we have $c = O(\sqrt{j})$.

This shows

Proposition 4.2. *The group $L^\times G$ maps to \mathbb{P}_0^{ind} and the inclusion is a morphism of ind-schemes. One gets an ind-scheme structure on X^{aff} by*

$$X^{aff} = \cup_n X_n^{aff} := \cup_n X^{aff} \cap \mathbb{P}_n^{ind}$$

5. BUNDLES ON CURVES

In this section discuss the connection between LG and the moduli stack $Bun_G(C)$ of principal G bundles on a curve C . We explain the connection between the embedding X_{ind}^{aff} and bundles on nodal curves as well as a connection between X_{ind}^{aff} and a construction of a completion of $Bun_G(C)$ on nodal curves due to Faltings.

5.1. The double coset construction. To begin with we make another addition to our list of important subgroups of the loop group. The subgroup is associated to the data of a fixed compact Riemann surface C and a point $p \in C$. The holomorphic version is defined as follows. Fix a local isomorphism z from $\{|z| > 1/2\}$ in $\infty \in \mathbb{P}^1(\mathbb{C})$ to a neighborhood around p . Set $D_p = \{|z| > 1\}$ and $C^* = C - \overline{D_p}$ so that $\overline{D_p} \cap \overline{C^*} = S^1$ identified with $\{|z| = 1\}$. Consider now the smooth loop group $L^{sm}G = C^\infty(S^1, G)$. Define

$$L_C^{sm}G = Hol(C^*, G) \subset L^{sm}G,$$

as the subgroup of loops that extend to give a holomorphic map $C^* \rightarrow G$. The algebraic version, $L_C G$, is the ind algebraic group which associates to a \mathbb{C} -algebra R the group

$$L_C G(R) := G((C - p)_R)$$

of algebraic maps $(C - p) \times_{\mathbb{C}} \text{Spec } R \rightarrow G$. When working algebraically we restrict to semi simple G ; this is explained after remark 21.

We have the following theorem proved by Atiyah:

Theorem 5.1. *The set of isomorphism classes of holomorphic principal G -bundles on C is equal to*

$$L_C^{sm}G \backslash L^{sm}G / L^{sm,+}G$$

Proof. [26, 8.11.5] □

Remark 21. One proves the result by noting a G -bundle on C^* is holomorphically trivial. Additionally the bundle is trivial on D_p and so the bundle is determined by a “transition function” $\gamma \in LG$; modding out by the two subgroups amount to accounting for changes of trivialization.

In the algebraic setting instead of looking at a small disc around $p \in C$ we look at a formal neighborhood $D_p \cong \text{Spec } \mathbb{C}[[z]]$. For a moment let us consider the case $G = SL_n$. In this case we can identify SL_n -bundles with vector bundles that have trivial determinant. In turn vector bundles we can identify with locally free sheaves. Over $C - p$ a locally free sheaf is just projective module E over the ring $A = \mathbb{C}[C - p]$ which is a Dedekind domain. According to the structure theorem for projective modules over a Dedekind domain we have that

$$E \cong A^{\oplus rk(E)-1} \oplus \det E = A^{\oplus rk(E)}.$$

For general simple G choose a faithful irreducible representation $G \subset SL(V)$ and identify a principal G -bundle with $SL(V)$ bundle together with a reduction of the structure group to G . Then over $C - p$ the $SL(V)$ bundle is trivial and the reduction to G is preserved so the G bundle is trivial on $C - p$. More generally, for a family of curves, we have the following result due to Drinfeld and Simpson [8].

Theorem 5.2. *Let S be a scheme and C a smooth proper scheme over S with connected geometric fibers of pure dimension 1 and let G be a semisimple group. Let D be a subscheme of C such that the projection $D \rightarrow S$ is an isomorphism. Set $U = C - D$. Then for any G -bundle F on C its restriction to U becomes trivial after a suitable faithfully flat base change $S' \rightarrow S$ with S' being locally of finite presentation over S . If S is a scheme over $\mathbb{Z}[n^{-1}]$ where n is the order of $\pi_1(G(\mathbb{C}))$ then S' can be chosen to be étale over S .*

This shows that for a fixed curve over \mathbb{C} a principal G -bundle is determined by a transition function over a punctured formal neighborhood of a point. This shows, as in the holomorphic case, there is a set theoretic bijection between a double coset space and the principal bundles on C . In fact more is true. Consider $L_C G \backslash LG / L^+ G$ as a stack by taking it to be the global quotient stack $[L_C G \backslash (LG / L^+ G)]$, then

Theorem 5.3. *There is a canonical isomorphism of stacks*

$$\text{Bun}_G(C) \cong L_C G \backslash LG / L^+ G$$

This was proved for $G = SL_n$ by Beauville and Lazlo and for general G by Lazlo and Sorger in [21]. In both cases we also have the presentation in terms of flag varieties for LG . Set $Y_C = L_C G \backslash LG$ and $Y_D = LG / L^+ G$ and consider the diagonal action of LG on $Y_C \times Y_D$, then

$$\text{Bun}_G(C) \cong \frac{Y_C \times Y_D}{LG}.$$

5.2. The connection with X_{ind}^{aff} and bundles on nodal curves. To begin this discussion consider the formal neighborhood of a node on a fixed nodal curve C . The neighborhood is isomorphic to

$$D = \text{Spec } \mathbb{C}[[x, y]] / xy$$

any bundle over this formal neighborhood is trivial. A change of trivialization is given by $\tau: D \rightarrow G$ which we can identify with

$$G(D) \leftrightarrow (\gamma_1, \gamma_2) \in G[[x]] \times G[[y]] \text{ with } \gamma_1(0) = \gamma_2(0)$$

In fact this space of trivializations is, in the notation of theorem 3.8, just $S(\{0\}) = G \ltimes (N^- \times N)$ where $N^\pm \subset G[[z^\pm]]$ are those loops that are the identity mod z^\pm . In other words we have identified an orbit

$$\text{Orb}(\{0\}) = G((z)) \times G((z^{-1})) / S(\{0\})$$

in X_{ind}^{aff} as parametrizing bundles on C , with a trivialization away from the node. The orbit $\text{Orb}(\{0\})$ is one of $r+1 = rk(G)+1$ orbits of ‘codimension 1.’ This terminology is justified because there is a natural correspondence between $T^\times \times T^\times$ orbits in \overline{T}^\times_0 and $G^\times((z)) \times G^\times((z^{-1}))$ orbits in X_{ind}^{aff} and $\text{Orb}(\{0\})$ corresponds to a codimension 1 orbit in \overline{T}_0 .

The question arises if the other codimension 1-orbits, or more generally if all of the other orbits in X_{ind}^{aff} parametrize bundle data. In order to go further in addressing this question we need to distinguish between principal G -bundles and \mathcal{G} -torsors.

In general, given a curve $C \rightarrow B$ and a sheaf of group \mathcal{G} on C we define a \mathcal{G} -torsor to be a sheaf of sets \mathcal{F} on $C \rightarrow B$ together with a right action of \mathcal{G} such that (1) there is an étale cover $\{C_i \rightarrow C\}$ such that $\mathcal{F}(C_i) \neq \emptyset$ and (2) the action map $\mathcal{G} \times \mathcal{F} \rightarrow \mathcal{F} \times \mathcal{F}$ is an isomorphism.

Given a principal bundle F on C we can consider the sheaf of groups \mathcal{G} represented by the scheme $G \times C$ and the sheaf \mathcal{F} represented by F and in this way associate to each principal G bundle a \mathcal{G} -torsor. In fact there is a perfect dictionary between \mathcal{G} -torsors and G -bundles whenever $\mathcal{G}(\hat{\mathcal{O}}_x) \cong G[[z]]$ for every point x where $\hat{\mathcal{O}}_x$ is the completion of the local ring at x .

Quasi parabolic bundles can also be interpreted as \mathcal{G} -torsors. Quasi parabolic bundles are G -bundles on a curve together with reductions to a parabolic subgroups P_i of G at a finite number of points x_1, \dots, x_m in the curve. These are equivalent to \mathcal{G} -torsors for a sheaf of groups $\mathcal{G} \subset G(\mathcal{O})$. More precisely these are \mathcal{G} torsors for which $\mathcal{G}(\hat{\mathcal{O}}_x)$ is generically the standard maximal parabolic subgroup $G[[z]]$ of LG but at each of the points x_i we have $\mathcal{G}(\hat{\mathcal{O}}_{x_i}) = P_i = \{\gamma \in G[[z]] \mid \gamma(0) \in P_i\}$. Using the language of \mathcal{G} -torsors we refer to quasi parabolic bundles as *\mathcal{G} -torsors with $\mathcal{G}(\hat{\mathcal{O}}_{x_i}) = P_i$* ; it is understood that for all other points $\mathcal{G}(\hat{\mathcal{O}}_x) \cong G[[z]]$.

To return to the question of interpreting the orbits of X_{ind}^{aff} we define a *parahoric subgroup* of $L^\times G$ to be any subgroup $\mathcal{P} \subset LG$ that is conjugate to one of the groups

$$\mathcal{P}_\eta^\times = \{\gamma \in LG \mid \lim_{s \rightarrow 0} \eta(s)\gamma\eta(s)^{-1} \text{ exists} \}$$

where η is a co-character such that the composition $\mathbb{C}^\times \xrightarrow{\eta} T^\times = \mathbb{C}^\times \times T \xrightarrow{P_1} \mathbb{C}^\times$ is given by a positive integer. These are really just parabolic bundles of $L^\times G$ but they are conventionally called parahoric subgroups. We use the notation \mathcal{P}_η to denote the quotient $\mathcal{P}_\eta^\times / \mathbb{C}^\times = \mathcal{P}_\eta^\times \cap LG$ and define *parahoric subgroups of LG* to be any subgroup conjugate to a \mathcal{P}_η . Let $\mathcal{K} = \mathbb{C}(C)$ be the constant sheaf of rational functions on C . Given a finite set of points x_1, \dots, x_m on a curve C and parahoric subgroups P_i of LG we define a *quasi parahoric bundle* as a \mathcal{G} -torsor for \mathcal{G} a sub sheaf of $G(\mathcal{K})$ with $\mathcal{G}(\hat{\mathcal{O}}_{x_i}) = P_i$. Quasi parahoric bundles as defined here seemed to be first discussed by Teleman, see [28, sect. 9]. They have recently received more attention by Heinloth [15] and by Balaji and Seshadri [31].

Proposition 5.4. *Let C be a nodal curve with a unique node x such that $C - x$ is affine. Then the orbit $O = \text{Orb}(\{j\}) \subset X_{ind}^{aff}$ parametrizes G bundles on C with $S(\{j\})$ -structure at the node. Let \tilde{C} be the normalization of C and let y, z be the preimages of x . Then the bundles parametrized by O correspond to parahoric bundles E on \tilde{C} with an isomorphism in a Levi subgroup $L = \mathcal{G}(\hat{\mathcal{O}}_z) \cap \mathcal{G}(\hat{\mathcal{O}}_y)$.*

Proof. The case $j = 0$ was explained in the beginning of this subsection. Set $J = \{j\}$; from theorem 3.8 we have

$$O = \frac{L^\times G \times L^\times G}{T(J) \cdot S(J)}$$

$$T(J) = \bigcap_{i \neq j} \ker \alpha_i \times \bigcap_{i \neq j} \ker \alpha_i$$

Choose a co-character ζ such that $\zeta(t) \in \bigcap_{i \neq j} \ker \alpha_i \cap T = \bigcap_{i \neq j, 0} \ker \alpha_i$ and $\alpha_j(\zeta(t)) = t^{\langle \zeta, \alpha_j \rangle} \neq 1$; this is always possible if $j \neq 0$. Then

$$(u, \zeta(t)) \xrightarrow{\alpha_0} u \prod_{i \geq 1} t^{\langle \zeta, \alpha_i \rangle} = ut^{\langle \zeta, \alpha_j \rangle} = ut^m$$

with $m \neq 0$ and thus the co-character

$$u \xrightarrow{\zeta'} (u, \zeta(u)^{-1/m})$$

satisfies $\zeta'(u) \times \zeta'(u) \subset T(J)$ and for dimensional reasons we have equality. This means the class of

$$(u, \gamma), (u', \gamma') \in L^\times G \times L^\times G$$

in O has a unique representative of the form $(1, h), (1, h')$ where the γ 's and h 's differ by multiplication by an element of $\bigcap_{i \neq j, 0} \ker \alpha_i$. In other words we have

$$O = \frac{L^\times G \times L^\times G}{T(J) \cdot S(J)} \cong \frac{LG \times LG}{S(J)} = \frac{LG \times LG}{\Delta(L_J) \ltimes (U_J^- \times U_J)}$$

This shows O parametrizes \mathcal{G} -torsors on C such that $\mathcal{G}(\hat{\mathcal{O}}_x) = S(J)$. More precisely, the uniformization theorem for G -bundles also holds for \mathcal{G} -torsors, see [15]. Therefore given a \mathcal{G} -torsor \mathcal{E} on C with $\mathcal{G}(\hat{\mathcal{O}}_x) =$

$S(J)$ it will be trivial in a formal neighborhood of x and on the complement of x and we can consider the data

$$(\mathcal{E}, C = C^* \cup_{D^*} D, \tau_{C^*}, \tau_D = (\tau_y, \tau_z))$$

where $D = \text{Spec } \hat{\mathcal{O}}_x \cong \text{Spec } \mathbb{C}[[y, z]]/yz$, $C^* = C - x$ is a smooth affine curve and $\tau_D: S(J) \rightarrow \mathcal{E}(D)$ is a trivialization which you can restrict to either side of the node to get τ_y or τ_z ; similarly $\tau_{C^*}: G(C^*) \rightarrow \mathcal{E}(C^*)$. From this data we get two loops $(\gamma_1 = \tau_{C^*}/\tau_y, \gamma_2 = \tau_{C^*}/\tau_z)$. Forgetting the trivialization τ_D is equivalent to identifying loops that differ by an element in $\mathcal{G}(D) = S(J)$. Therefore the space of all tuples

$$(C = C^* \cup_{D^*} D, \mathcal{E}, \tau_{C^*})$$

is represented by $\frac{LG \times LG}{S(J)}$. Equivalently we can consider the data on the normalization:

$$(\tilde{C} = C^* \cup_{\tilde{D}^*} \tilde{D}, \tilde{\mathcal{E}}, \tau_{C^*}, \tau_x: \tilde{\mathcal{E}}(k(y)) \rightarrow \tilde{\mathcal{E}}(k(z)))$$

where $k(y), k(z)$ are the residue fields of $y, z \in \tilde{C}$. The equivalence arises because a trivialization of $\tilde{\mathcal{E}}$ over \tilde{D} respecting τ_x is equivalent to a trivialization of \mathcal{E} over D . To interpret this in term of the geometry of the orbit O note we have a natural map

$$O \rightarrow LG/P_J^- \times LG/P_J$$

with fiber L_J . This shows a point of O is equivalent to the data of $(g, \gamma_1, \gamma_2) \in L_J \times LG/P_J^- \times LG/P_J$ which give \mathcal{G} -torsors on \tilde{C} with $\mathcal{G}(\hat{\mathcal{O}}_y) = P_J^-, \mathcal{G}(\hat{\mathcal{O}}_z) = P_J$ together with $g \in \mathcal{G}(\hat{\mathcal{O}}_y) \cap \mathcal{G}(\hat{\mathcal{O}}_z) = \mathcal{G}(k(x))$. \square

Remark 22. The question of the modular interpretation of the remaining, higher codimensional orbits requires more care. There are some results in this direction when one restricts to the divisor $\overline{Orb}(\{0\})$ which generically corresponds to the standard maximal parabolic subgroup $G[[z]]$. In this case higher codimensional orbits have been interpreted as torsion free sheaves for GL_n and Sp_n . The other approach is to consider bundles on modifications of nodal curves as in [29, 17, 18]. In the analytic setting one can nevertheless fit together all the orbits of X^{aff} into a complex analytic space that serves as a completion of bundles over nodal curves in families. The appropriate algebraic analogue is work in progress.

Even without a complete modular interpretation of X_{ind}^{aff} the previous proposition already tells us something interesting. Given that bundles on a nodal curve are equivalent to bundles on the normalization together with a ‘transition function’ $\in G$ at the node, it is a natural first guess to try to complete $Bun_G(C)$ simply by compactifying G . However the previous proposition shows that this is not sufficient; namely, in families a principal bundle may develop parahoric structure at the node. Figure 3 illustrates this; compactifying G only tells you about the divisor D_0 .

We now give a couple of example of parahoric subgroups and parahoric bundles that can develop in families. We treat first the case $G = SL_2$. The standard parahoric subgroups of LG are

$$\mathcal{P}_0 = G[[z]]$$

$$\mathcal{P}_1 := \left\{ \gamma(z) = \begin{pmatrix} a & b/z \\ cz & d \end{pmatrix} + \begin{pmatrix} 0 & b_0 \\ 0 & 0 \end{pmatrix} + z \begin{pmatrix} a_1 & b_1 \\ 0 & d_1 \end{pmatrix} + z^2 \begin{pmatrix} a_2 & b_2 \\ c_2 & d_2 \end{pmatrix} + \dots \mid \det \gamma(z) = 1 \right\}$$

The first corresponds to the co-character $u \mapsto (u, \text{diag}(1, 1)) \in T^\vee$ and the second corresponds to $(u, \text{diag}(\sqrt{u}, \frac{1}{\sqrt{u}}))$.

In the case of SL_r , all the maximal parahoric bundles are conjugate to $G[[z]]$ by outer automorphisms which can either be interpreted as ‘fractional’ loops in G or as honest loops in GL_r . For example

$$\begin{pmatrix} 1 & 0 \\ 0 & z^{-1} \end{pmatrix} \mathcal{P}_1 \begin{pmatrix} 1 & 0 \\ 0 & z \end{pmatrix} = \begin{pmatrix} z^{1/2} & 0 \\ 0 & z^{-1/2} \end{pmatrix} \mathcal{P}_1 \begin{pmatrix} z^{-1/2} & 0 \\ 0 & z^{1/2} \end{pmatrix} = \mathcal{P}_0$$

As a result all \mathcal{G} -torsors with \mathcal{P}_i structure always have interpretations in terms of vector bundles. In the case at hand if E is the trivial rank 2 vector bundle over $D = \text{Spec } \mathbb{C}[[z]]$ and s_1, s_2 are two non vanishing sections then any other such sections can be obtained as $\gamma.s_1, \gamma.s_2$ with $\gamma \in \mathcal{P}_0$. On the other hand if we require s_1 to have a pole at the closed point and s_2 to be non vanishing at the closed point then any other such sections can be obtained as $\gamma.s_1, \gamma.s_2$ with $\gamma \in \mathcal{P}_1$. The latter case corresponds to the vector bundle $\mathcal{O}_D \oplus \mathcal{O}_D(1)$ over D . In other words, if we have a smooth curve C and a point p then $G(C - p) \backslash LG/\mathcal{P}_0$ is

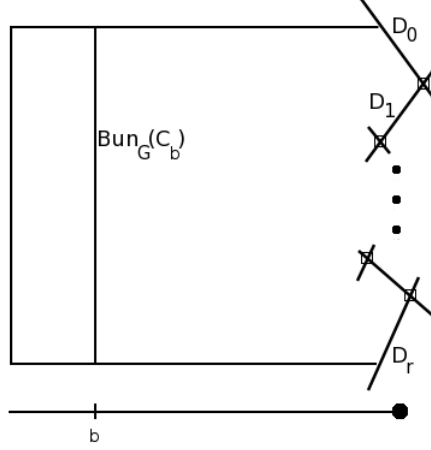


FIGURE 3. Schematic of $\overline{Bun_G(C/B)}$ over the base B . The general fiber is just $Bun_G(C_b)$ for the smooth curve C_b . Over the special fiber of B we get a divisor with normal crossings and $r + 1$ components. The divisor D_i has a dense open subset, the complement of the squares, which corresponds to $Orb(\{i\})$.

the moduli space of rank 2 vector bundles with trivial determinant while $G(C - p) \backslash LG/\mathcal{P}_1$ corresponds to those vector bundles with determinant $\mathcal{O}_C(p)$.

To bring in nodal curves consider the curve $C = Bl_{(0,0)}(\mathbb{P} \times \mathbb{A}^1)$ over \mathbb{A}^1 . The general fiber is \mathbb{P}^1 and the special fiber is $\mathbb{P}^1 \cup \mathbb{P}^1$. Possibly the simplest example of parahoric structure developing can be seen with the bundle $\mathcal{O}_C \oplus \mathcal{O}_C(E)$ where E is the exceptional divisor of the blow up. In this case we get the trivial rank 2 bundle on all the smooth fibers while on the special fiber we get $\mathcal{O} \oplus \mathcal{O}(-1)$ on the exceptional divisor and $\mathcal{O} \oplus \mathcal{O}(1)$ on the other \mathbb{P}^1 .

On the other hand $\mathcal{O} \oplus \mathcal{O}(-1)$ is a GL_2 bundle on C . If we work with honest SL_2 bundles on C we still see the parahoric structure at the node, but we are forced to get an SL_2 bundle on the special fiber as well so the parahoric structure at the node is cancelled by opposite parahoric structure at other points of the curve. For example consider the SL_2 bundle on C obtained by inducing the \mathbb{C}^\times -bundle $\mathcal{O}(E)$ via $\mathbb{C}^\times \rightarrow T \subset SL_2$, $t \mapsto \text{diag}(t, 1/t)$. Then the restriction of the special fiber is $L = \mathcal{O}(-1) \oplus \mathcal{O}(1) \cong \mathcal{O}(-p) \oplus \mathcal{O}(q)$ on E where p, q are zero and infinity on \mathbb{P}^1 . On the other \mathbb{P}^1 the bundle is L^\vee .

This example generalizes completely to any 1 dimensional family of curves $C \rightarrow B$ degenerating to a reducible nodal curve and to any simple group. Let C_1, C_2 be the components of the special fiber so $\mathcal{O}_C(C_1)|_{C_1} \cong \mathcal{O}_{C_1}(-y)$ and $\mathcal{O}_C(C_1)|_{C_2} \cong \mathcal{O}_{C_2}(-z)$ where y, z are the preimages of the node x in the normalization of the special fiber of C . Choose a co-character η giving maximal parahoric subgroup $\mathcal{P} \neq G[[z]]$. Now choose an irreducible representation $G \subset SL(V)$ and consider the G bundle $P = \mathcal{O}_C(C_1) \times_\eta G$. The representations V allows us to give an isomorphism

$$P \cong \oplus_{i=1}^n \mathcal{O}_C(a_i C_1) \oplus_{j=1}^m \mathcal{O}_C(-b_j C_1) \oplus \mathcal{O}_C^{\oplus \dim(V) - n - m}$$

where $a_i, b_j > 0$ are integers. Consider also the vector bundle

$$P' = \oplus_{i=1}^n \mathcal{O}_C(a_i C_1) \oplus \mathcal{O}_C^{\oplus \dim(V) - n}$$

then for every smooth fiber C_b the restriction of P' admits a reduction of the structure group to G while restriction to the special fiber has parahoric structure given by \mathcal{P} at the node and has no reduction to G . The bundle P restricts to a G -bundle on every fiber. It has parahoric structure at the node as well as complementary parahoric structure at the points of $\text{div}(y) \cap (C_1 - y) \cup \text{div}(z) \cap (C_2 - z)$ where by abuse of notation we use y, z to denote both a point in C_i as well as a local coordinate for this point.

We finish with one final example of a parahoric subgroup of SO_5 which shows, unlike the SL_r case, the parahoric structures that appear cannot always be interpreted as twists of the standard structure $G[[z]]$. It will be enough to work with the Lie algebra which we present as

$$\mathfrak{so}_5 = \left\{ \begin{bmatrix} a_1 & a_2 & 0 & b & -h_1 \\ a_3 & a_4 & -b & 0 & -h_2 \\ 0 & c & -a_1 & -a_3 & -g_1 \\ -c & 0 & -a_2 & -a_4 & -g_2 \\ g_1 & g_2 & h_1 & h_2 & 0 \end{bmatrix} \mid a_i, b, c, g_i, h_i \in \mathbb{C} \right\}$$

the Lie algebra of a maximal torus is given by diagonal matrices and more generally we can pick out the root spaces

$$\begin{aligned} (X_1, Y_1) &\leftrightarrow (a_2, a_3) \\ (X_2, Y_2) &\leftrightarrow (\sqrt{2}h_2, -\sqrt{2}g_2) \\ (X_3, Y_3) &\leftrightarrow (\sqrt{2}h_1, -\sqrt{2}g_1) \\ (X_4, Y_4) &\leftrightarrow (b, c) \end{aligned}$$

For example

$$X_1 = \begin{bmatrix} & 1 & & & \\ & & & & \\ & & & & \\ & & & -1 & \\ & & & & \end{bmatrix} \quad Y_1 = \begin{bmatrix} & 1 & & & \\ & & & & \\ & & & & \\ & & & -1 & \\ & & & & \end{bmatrix}$$

The maximal parahoric subgroups are given by the vertices of the Weyl alcove $\subset \mathfrak{t}$. Identify $\mathfrak{t} = \mathbb{C}^2$ using the basis $H_1 = [X_1, Y_1]$ and $H_2 = [X_2, Y_2]$. Then the Weyl alcove is a triangle with supporting hyperplanes

$$y = x, \quad y = x/2, \quad y = 1/2$$

The vertices are $(0, 0)$, $(1/2, 1/2)$ and $(1, 1/2)$. The first corresponds to $G[[z]]$. The third corresponds a subgroup with a Levi factor $\cong SO_5$ and it is conjugate to $G[[z]]$ by an outer automorphism. The parahoric corresponding to $(1/2, 1/2)$ has a Levi factor isomorphic to $SL_2 \times SL_2$ the Lie algebra of this Levi is given by

$$\mathfrak{sl}_2 \times \mathfrak{sl}_2 \cong \begin{pmatrix} H_4 & Y_4 \otimes z \\ X_4 \otimes z^{-1} & -H_4 \end{pmatrix} \times \begin{pmatrix} H_1 & X_1 \\ Y_1 & -H_1 \end{pmatrix}$$

It is conjugate by outer automorphisms to the $SL_2 \times SL_2 \subset SO_5$ whose Lie algebra is

$$\begin{pmatrix} H_4 & X_4 \\ Y_4 & -H_4 \end{pmatrix} \times \begin{pmatrix} H_1 & X_1 \\ Y_1 & -H_1 \end{pmatrix}$$

The non isomorphic Levi subgroups occur because of the different length roots in \mathfrak{so}_5 . This demonstrates that when a node develops parahoric structure associated to the co-character $(1/2, 1/2)$ one can get not an SO_5 bundle at the node but a bundle with a reduction to $SL_2 \times SL_2$ at the node.

5.3. Formal Construction. The motivation to construct the embedding X_{ind}^{aff} came from a formal algebraic construction of Faltings given in [10, 6] and [9, 3]. Given a family of smooth curves $C \rightarrow B$ degenerating to a nodal special fiber C_0 , Faltings constructs a formal algebraic stack \mathcal{F} over B which contains as a dense open substack the moduli stack $Bun_G(C/B)$. Faltings proves the stack \mathcal{F} is complete (see [9, 3.1]).

We will not give here the details of the construction of \mathcal{F} but the key ingredient in the construction is a groupoid F over the formal disc $D = \text{Spec } \mathbb{C}[[s]]$ which Faltings uses as a local model for G -bundles near a node. We explain the connection between the groupoid F and X_{ind}^{aff} .

The construction of F depends heavily on the toric variety \overline{T}^\vee and $\overline{T}_0^\vee \cong \mathbb{A}^{r+1}$ of section 3.2. To define the groupoid we need a certain subgroup scheme $G^\Delta \subset \overline{T} \times L_x G \times L_y G$ where $L_x G = G((x))$ and

similarly for y . Over a point $p \in \overline{T}$ the subgroup $G_p^\Delta \subset L_x G \times L_y G$ is given as follows. If $p \in \overline{T^\times}$ is written as $t.\eta(0)$ where $\eta \in V_{T^\times}$ then the group G_p^Δ is isomorphic to

$$G_p^\Delta \cong \Delta(L_\eta) \ltimes (U_\eta^- \times U_\eta)$$

where $P_\eta := L_\eta U_\eta \subset G^\times((x))$ and $P_\eta^- := L_\eta U_\eta^- \subset G^\times((y))$ are the parabolic or parahoric subgroup associated to η . That such a group scheme exists is proven in [10, 6.3].

The groupoid $F = (F_1 \rightrightarrows F_0)$ is given as

$$\begin{aligned} F_0 &= \overline{T^\times} \times L_x G \times L_y G \\ F_1 &\subset F_0 \times W^{aff} \ltimes T^\times \times G^\Delta \end{aligned}$$

The arrows $F_1 \rightrightarrows F_0$ are given by projection and an action map. More precisely, if we denote a point of F_0 as (p, γ) and a point of $W^{aff} \ltimes T^\times \times G^\Delta$ as (w, t, g_p^Δ) where for every $p \in \overline{T^\times}$ we have $g_p^\Delta \in G_p^\Delta$, then the action is

$$\begin{aligned} (1, 1, g_p^\Delta).(p, \gamma) &= (p, g_p^\Delta \gamma) \\ (w, t, 1).(p, \gamma) &= (w(t.p)w^{-1}, \gamma) \end{aligned}$$

The meaning of $F_1 \subset F_0 \times W^{aff} \ltimes T^\times \times G^\Delta$ is that in order for the action map to make sense we only consider those points $((p, \gamma), (w, t, g_p^\Delta))$.

Because, in the notation of section 3.2, we have $\overline{T^\times}/W^{aff} = \overline{T_0^\times}$ it follows that F is isomorphic to the groupoid F' :

$$\begin{aligned} F'_0 &= \overline{T_0^\times} \times L_x G \times L_y G \\ F'_1 &= F'_0 \times T^\times \times G^\Delta. \end{aligned}$$

Comparing with proposition 3.6 it follows that the coarse moduli space of the associated stack of F' is isomorphic to X_{ind}^{aff} pulled back to $\text{Spec } \mathbb{C}[[s]]$ via the inclusion $\mathbb{C}[s] \subset \mathbb{C}[[s]]$.

Faltings construction gives a formal completion of bundles on nodal curves. As we have already mentioned in remark 22 in the analytic setting there is no trouble constructing this completion as an honest complex analytic space. Constructing an honest algebraic stack will be done in a follow up paper.

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